Generalized S-closed spaces and strongly S-closed spaces

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Abstract

This paper discusses both of S-closed spaces, strongly S-closed spaces and properties of them as well as the effect of some functions on them. This study focuses on show the relation between these spaces and the spaces that have closed cover.

Key words : S-closed spaces, strongly S-closed spaces, Mapping space

الملخص

يناقش هذا البحث كلا من الفضاءات المغلقة من النمط S و الفضاءات المغلقة القوية من النمط S وخصائصها بالإضافة إلى تأثير بعض الدوال عليها. وكذلك تركز هذه الدراسة على إظهار العلاقة بين هذه الفضاءات و الفضاءات ذات الغطاء المغلق.

1 INTROCUCTION

To begin some basic definitions and important results that it is need to show new results in this study. One of the definitions is given by, if $A \subset X$ Let X be a topological space, then, the closure of A is the intersection of all closed sets in X, containing A, it is denoted by $\bar{A}$ or cl(A). $\bar{A} = \cap \{F: A \subset F \text{ and } F \text{ is closed}\}$ [1]. Then, it is established the theorem if A and B be two subsets of a space X, then. (i) $\bar{A}$ is closed set, (ii) A is closed if $A = \bar{A}$, (iii) $\bar{\emptyset} = \emptyset$, (iv) $A \subset \bar{A}$, (v) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and (vi) $\bar{\bar{A}} = \bar{A}$ and note that if $\bar{A} = X$, then A is called dense in X for the proof and more details see [2].

By studying the basic definition in general topology, it is easily to defined the S-closed space as, if a topological space X, is called S-closed space if for every semi- open cover $\{u_i: i \in I\}$ of X, there exists a finite subset I_0 of I such that; $X = \cup \{\bar{U}_i: i \in I_0\}$ for more details see [1], [2], [3], [4], [5] and [6].

This paper draws on and derives support from the studies mentioned above and show the relation between S-closed spaces and strongly S-closed spaces. In particular, this paper investigates a closed spaces, subspaces and strongly closed space. Also, this paper discusses the relationship between these spaces. Finally, the paper draws conclusions and offers suggestions for future research.

2. S-closed space

Definition 2.1:

A topological space X, is called S-closed space if for every semi- open cover $\{u_i: i \in I\}$ of X, there exists a finite subset I_0 of I such that; $X = \cup \{\bar{U}_i: i \in I_0\}$ [6]. According to the study [6].

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Theorem 2.1.

A space X is S-closed if and only if every regular closed cover of X has a finite subcover, see [6].

Theorem 2.2.

A topological space X. If X is S-closed space, then any preopen cover of X has a finite subcover, see [7].

Theorem 2.3.

If X is S-closed space then any regular open cover of X has a finite subcover.

Proof: -

Let $\{u_i : i \in I\}$ be a regular open cover of X. $u_i = (\overline{U_\iota})^o$, then for every $i \in I$, it is obtained that $\overline{u_\iota} \subseteq \overline{(\overline{u_\iota})^o} \Rightarrow u_i \subseteq (\overline{u_\iota})^o$ and thus, is semi-open cover of X, which is S-closed. $\{\overline{u_\iota} : i \in I\}$. So that

$$X = \bigcup_{i \in I_o} u_i$$

Let X be a topological space such that u^o is a cover of X. Then, if any preopen cover of X has a finite subcover then X is S-closed space. To prove this fact, assume that $\{u_i : i \in I\}$ be a semi-open cover of X. for any $i \in I$, $u_i \subseteq \left(\overline{u_\mathrm{i}^o}\right)$ by definition of semi-open sets is semi-open set of X. $u_\mathrm{i}^o \subseteq \left(\left(\overline{u_\mathrm{i}^o}\right)\right)^o$ then u_i^o, so $\{u_\mathrm{i}^o : i \in I\}$ is a preopen cover of X. By hypothesis, there exists a finite subfamily I_o of I such that; since $\left(\overline{u_\mathrm{i}^o}\right) \subseteq (\overline{u_\iota})$, then,

$$X = \bigcup_{i \in I} \overline{u_\iota}$$

and X is S-closed.

Theorem 2.4.

A topological space X is S-closed space, iff for each family $\{u_i : i \in I\}$ there exists a finite

$$\bigcap_{i \in I} u_i = \emptyset$$

semi-closed subsets of X for which subfamily I_o of I Such that;

$$\bigcap_{i \in I} u_\mathrm{i}^o = \emptyset$$

For the proof see [8].

Theorem 2.5.

A topological space X is S-closed iff

$$\bigcap_{i \in I} u_i \neq \emptyset$$

Where $\{u_i : i \in I\}$ is a family of semi-closed subsets of X. For which

$$\bigcap_{i \in I} u_\mathrm{i}^o \neq \emptyset$$

for a finite subfamily I_o of I, see [6] and [8].

2.1 LOCALLY AND COUNTABLE S-CLOSED SPACES

X is locally S-closed space if each point of X has a neighbourhood which is S-closed relative to X. There is theorem is said that for a space X, the following statements are equivalent

(i) X is locally S-closed.

(ii) Each point of X has an open neighbourhood which is S-closed relative to X is v such that $\bar{v}$

(iii) Each point of X has an open neighbourhood S- closed relative to X. For the proof see [5].

Also, every S-closed space is countably S-closed space for the proof see [5].

2.2 S-CLOSED SUBSPACES

A subset A of a space X is said to be S-closed relative to X if for every cover $\{u_\alpha : \alpha \in I\}$ of A by semi-open sets of X. There exists a finite subset I_o of I such that $A \subset\cup \{\overline{u_\alpha} : \alpha \in I_o\}$.

Theorem 2.6.

A semi closed and open subset of an S-closed space X is an S-closed subspace of X.

Proof: -

Let A be a semi-closed and open subset of an S-closed X. We have to show that A is an S-closed relative to X. Suppose $\{u_i : i \in I\}$ is a semi-open cover of A.

$$A \subset \bigcup_{i \in I} u_i$$

Then,

$$A^c \cup A = A^c \cup \left(\bigcup_{i \in I} u_i \right) = X$$

$\{A^c, u_i : i \in I\}$, is a semi-open cover of X. Since X is S-closed. There exists a finite subfamily I_o of I such that;

$$X = \overline{A^c} \cup \left(\bigcup_{i \in I} \overline{u_\iota} \right) = X$$

Since A is open.

$$X = A^c \cup A = A^c \cup \left(\bigcup_{i \in I} \overline{u_\iota} \right)$$

$A^o = A$, thus

$$A \subset \bigcup_{i \in I} \overline{u_\iota},$$

hence A is S-closed relative to X.

Theorem 2.7.

If there exists a dense subset G of X which is S-closed relative to X. Then X is S-closed, see [7].

Theorem 2.8.

If a function $f: X \longrightarrow Y$ is continuous almost-open and G is S-closed relative to X. Then $f(G)$ is S-closed relative to Y, see [3].

2.3 MAPPINGS OF AN S-CLOSED SPACE

There is here important theorem is said that, if a mapping $f: X \longrightarrow Y$ is continuous and X is S-closed, then $f(X)$ is S-closed. To prove, let $\{u_i: i \in I\}$ be semi-open cover of $f(X)$. Since, f is continuous. Then $\{f^{-1}(u_i): i \in I\}$ is semi-open cover of X. Since X is S-closed space. Then there exists a finite subfamily I_o of I such that;

$$X = \cup \{\overline{f^{-1}(u_i)}: i \in I_o\}$$

$$\overline{f^{-1}(u_i)} \subset f^{-1}(\overline{u_i})$$

Since f is continuous, then,

$$X = \cup \{\overline{f^{-1}(u_i)}: i \in I_o\}$$

$$f(X) = f\left(\cup \left\{\overline{f^{-1}(u_i)}: i \in I_o\right\}\right) = \cup\ (\{ff^{-1}(\overline{u_i}): i \in I_o\}) \subset = \cup \{\overline{u_i}: i \in I_o\}$$

Then, $f(X)$ is S-closed space.

Theorem 2.9.

If $f: X \longrightarrow Y$ is continuous irresolute surjection from an S-closed space X. Then Y is S-closed, see [6] and [7].

3 STRONGLY S-CLOSED SPACE

Definition 3.1.

A topological space X is called strongly S-closed if every closed cover of X has a finite subcover [9].

Theorem 3.1.

A space X is strongly S-closed space iff X has a finite dense subset, see [9].

Theorem 3.2.

If X is strongly S-closed space. And A is an open set in X, then A is strongly S-closed. To show that, let us assume X is strongly S-closed space. A any open set in X.

By supposing $\lambda = \{G_\alpha\}_{\alpha \in \lambda}$ is a cover of A by closed sets in X. Closed cover of X. $\lambda^* = \lambda \cup \{A^c\}$. Since, X is strongly S-closed space. Then, there exists a finite subcover $\{G_{\alpha 1}, G_{\alpha 2}, \dots, G_{\alpha n}, A^c\}$ from λ^*. Where $X = G_{\alpha 1} \cup G_{\alpha 2} \cup \dots \cup G_{\alpha n} \cup A^c$. Then $\{G_{\alpha 1}, G_{\alpha 2}, \dots, G_{\alpha n}, A^c\}$ is a finite subcover of A from λ. Hence, A is strongly S-closed space. Whereas, If X is strongly S-closed space. Then every regular closed cover of X has a finite subcover. The family $\{A_\alpha: \alpha \in I\}$ has the finite intersection property iff every finite nonempty subcollection $A_{\alpha 1}, A_{\alpha 2}, \dots, A_{\alpha n}$ of $\{A_\alpha: \alpha \in I\}$ has the property that

$$\bigcap_{k=1}^{n} A_{\alpha k} \neq \emptyset,$$

for the proof see [9].

Definition 3.2.

The space X is strongly S-closed space iff every family $\{A_\alpha : \alpha \in I\}$ of open sets having the finite intersection property has a non-empty intersection [1] and [9].

To show this claim, firstly, let X be a strongly S-closed and $\{A_\alpha : \alpha \in I\}$ be a family of open sets. Having the finite intersection property. We have to show that

$$\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$$

Suppose that

$$\bigcap_{\alpha\in I} A_\alpha = \emptyset$$

Then, $\{X\backslash A_\alpha : \alpha \in I\}$ is a collection of closed sets. And

$$\cup\, X\backslash A_\alpha = X\backslash \bigcap_{\alpha\in I} A_\alpha = X\backslash\emptyset = X$$

Since X is strongly S-closed space. Then the closed cover $\{X\backslash A_\alpha : \alpha \in I\}$ has a finite subcover $X\backslash A_{\alpha 1}, A_{\alpha 2}, \dots A_{\alpha n}$. i.e

$$\bigcup_{i=1}^{n} X\backslash A_{\alpha i} = X \Rightarrow X\backslash \bigcap_{i=1}^{n} A_{\alpha i} = X$$

Then,

$$\bigcap_{i=1}^{n} A_{\alpha i} = \emptyset \ \mathit{Contradiction} \ \Rightarrow \bigcap_{i=1}^{n} A_{\alpha i} \neq \emptyset$$

Conversely; Take the hypothesis that every family of open sets in X having the finite intersection property has a non-empty intersection. We have to show that X is strongly S-closed. Let $\{F_\alpha : \alpha \in I\}$ be any closed cover of X. Then, $\{X\backslash F_\alpha : \alpha \in I\}$ is a family of open sets. Such that

$$\bigcap \{X\backslash F_\alpha\} = X\backslash\{\bigcup_{\alpha\in I} F_\alpha\} = X\backslash X = \emptyset$$

Consequently, hypothesis implies that the collection $\{X\backslash F_\alpha : \alpha \in I\}$ does not have the finite intersection property. Therefore, there is some finite subcollection $\{X\backslash F_{\alpha 1}, X\backslash F_{\alpha 2}, \dots, X\backslash F_{\alpha k}\}$ such that;

$$\left(\bigcap (X\backslash F_\alpha)\right)^c = X$$

$$\bigcup_{k=1}^{n} F_{\alpha k} = \bigcup_{k=1}^{n} \left(X\backslash (X \bigcup F_{\alpha k})\right)$$

Hence,

$$X = X\backslash\bigcap_{k=1}^{n}(X\backslash F_{\alpha k})$$

Thus, $F_{\alpha 1}$, $F_{\alpha 2}$,…, $F_{\alpha k}$ is a finite subcover of $\{F_{\alpha}: \alpha \in I\}$ for X. Then X is strongly S-closed.

Theorem 3.3.

The union of finite collection of strongly S-closed subsets of X is strongly S-closed, for the proof see [9].

Theorem 3.4.

A space X is countably strongly S-closed iff every countable closed cover of X has a finite subcover for X. Every strongly S-closed space X is also countably strongly S-closed, for the proof see [9].

Proof: -

Let $\{u_i: i \in I\}$ be any countable closed cover of X. Since, X is strongly S-closed. Then every closed cover has a finite subcover for X. And, in particular $\{u_i: i \in I\}$ has a finite subcover for X.
Implying X is countable strongly S-closed.

3.1 IMAGE OF STRONGLY S-CLOSED SPACE UNDER A CONTINUOUS MAPPING

Claim 1: If X is strongly S-closed space, and $f: X \longrightarrow Y$ is continuous function. Then $f(X)$ is strongly S-closed. To show this claim:

If λ is a cover of $f(X)$ by closed sets in Y. Then, $\{f^{-1}(G): G \in \lambda\}$ covers X by closed sets in X. Since, X is strongly S-closed. Then, $\{f^{-1}(G_1), f^{-1}(G_2), \dots, f^{-1}(G_n)\}$ covers X. The closed subcover of $f(X) \subseteq \lambda$. Then, $\{G_1, G_2, \dots, G_n\}$. Then, $f(X)$ is strongly S-closed space.

Claim 2: Y is continuous and X is strongly S-closed space, then Y is $\xrightarrow{onto}$ If $f: X$ strongly S-closed. By using claim 1, it is easy to prove it.

4 THE RELATION BETWEEN S-CLOSED SPACES AND STRONGLY S-CLOSED SPACES

According to section 2 and 3, it is clear to claim that: Every strongly S-closed space is S-closed. To prove, let $\{u_i: i \in I\}$ be regular closed cover of X. Since every regular closed set is closed. Then, $\{u_i: i \in I\}$ is closed cover of X. But, X is strongly S-closed. Then, $\{u_i: i \in I\}$ has a finite subcover. Hence X is S-closed space.

Theorem 4.1.

If $f: X \longrightarrow Y$ is completely continuous surjection and X is S-closed space. Then, Y is strongly S-closed, see [10].

Theorem 4.2.

Hausdorff S-closed space is strongly S-closed, see [5].

Theorem 4.3.

Let X be a finite space. If X is T1-space then, X is strongly S-closed, for the proof see [11].

4.1 THE RELATION BETWEEN S-CLOSED SPACE, STRONGLY S-CLOSED SPACE AND COMPACT SPACE

Theorem 4.4.

If a regular space X is S-closed space, then X is compact, for the proof see [5] and [12].

Theorem 4.5.

If $f: X \longrightarrow Y$ is SR-continuous surjection, and X is an S-closed space. Then Y is compact and also, if $f: X \longrightarrow Y$ is contra-continuous surjective, and X is compact space. Then Y is strongly S-closed.

Each extremally disconnected compact space is S-closed space. To show this assumption, let $\{u_i: i \in I\}$ be any regular closed cover of X. X is extremally disconnected. Then, any regular closed subset is open. Then, $\{u_i: i \in I\}$ is an open cover of X. X is compact. Then, $\{u_i: i \in I\}$ has a finite subcover. Hence X is S-closed space, for the proof see [11].

4.2 SOME OTHER RELATIONS

Every S-closed space is H-closed space. To show that, let $\{u_i: i \in I\}$ be an open cover of X. Since, any open set is semi-open set. Then, $\{u_i: i \in I\}$ is semi-open cover of X. X is S-closed space. There exists a finite subfamily I_o of I such that; $X = \cup \{\overline{u_\iota}: i \in I_o\}$.
Hence, X is H-closed space.

If a mapping $f: X \longrightarrow Y$ is an open bijection and Y is S-closed. Then, X is H-closed. To prove, let $\{u_i: i \in I\}$ be an open cover of X. f is an open bijection mapping. Then, $\{f(u_i): i \in I\}$ is an open cover of Y. Since, any open set is semi-open set. Then, $\{f(u_i): i \in I\}$ is semi-open cover of Y. Y is S-closed space. Then, there exists finite subfamily I_o of I such that; $Y = \cup \left\{\overline{f(u_\iota)}: i \in I_o\right\}$. Since f is bijection mapping. It is known that, $\overline{f(u_\iota)} = f(\overline{u_\iota})$,Then, $Y = \cup \{f(\overline{u_\iota}): i \in I_o\}, \Rightarrow f^{-1}(Y) = f^{-1}(= \cup \{f(\overline{u_\iota}): i \in I_o\}) \Rightarrow X = \cup$ $\{\overline{u_\iota}: i \in I_o\}$,Then X is H-closed space.

Claim 3: An extremally disconnected H-closed space is S-closed. To prove it, let $\{u_i: i \in I\}$ be regular closed cover of X. Any regular closed set is closed. Then, $\{u_i: i \in I\}$ is closed cover of X. Since X is extremally disconnected. Then, $\{u_i: i \in I\}$ is open cover of X. X is H-closed space. Then, there exists a finite subfamily I_o of I such that; $X = \cup \{\overline{u_\iota}: i \in I_o\}$, Since u_i is closed set

then, $\overline{u_\iota} = u_i \ \forall i \in I$. Then, $\{u_i: i \in I\}$ has a finite subcover of X. Hence, X is S-closed.

Any s-closed space is S-closed. To show this, let $\{u_i: i \in I\}$ be a regular closed cover. Since, any regular closed set is semi-regular. Then, $\{u_i: i \in I\}$ is semi-regular cover of X. X is s-closed space. Then, $\{u_i: i \in I\}$ has a finite subcover. Hence, X is S-closed space.

Let $f: X \longrightarrow Y$ be a surjection, and f is SR-continuous if X is s-closed space. Then, Y is strongly S-closed. To prove that, let $\{u_i: i \in I\}$ be any closed cover of Y. Then, $\{(Y \backslash u_i): i \in I\}$ is an open cover of Y. f is SR-continuous. Then, $\{f^{-1}(Y \backslash u_i): i \in I\}$ is semi-regular cover of X. X is s-closed. Then, $\{f^{-1}(Y \backslash u_i): i \in I\}$ has a finite subcover, $\{f^{-1}(Y \backslash u_i): i \in I_o\}$. Such that; $X = \cup \{(Y \backslash u_i): i \in I_o\}$. Then, $\{u_i: i \in I\}$ has a finite subcover. Hence Y is strongly S-closed space.

Claim 4: If X is strongly S-closed space. Then, X is mildly compact. To show that, suppose $\{u_i: i \in I\}$ be clopen (closed and open) cover of X. Since every clopen set is closed. Then, $\{u_i: i \in I\}$ is closed cover of X. X is strongly S-closed. Then $\{u_i: i \in I\}$ has finite subcover. Hence, X is mildly compact.

Claim 5: Let $f: X \longrightarrow Y$ surjection, if f is perfectly continuous and X mildly compact. Then Y is strongly S-closed. To prove this claim; let $\{u_i: i \in I\}$ be a closed cover of Y. Since f is perfectly continuous.

$\{f^{-1}(u_i): i \in I\}$ is clopen cover of X. X is mildly compact. Then, there exists a finite subfamily I_o of I such that;

$$X = \cup \{f^{-1}(u_i): i \in I_o\} \Rightarrow$$

$$f(X) = f(\cup \{f^{-1}(u_i): i \in I_o\}) \Rightarrow$$

$$Y = \cup f\{f^{-1}(u_i): i \in I_o \Rightarrow$$

$$Y = \cup \{u_i: i \in I_o\}$$

Then, Y is strongly S-closed space.

5. CONCLUSION

In this research were studied spaces stacked with a closed cover both S- closed space and strongly S- closed space. The study deals with the S- closed space and the strongly S-space in some detail in terms of properties and the effect of functions on it. The study also dealt with the relationships that bind these spaces together. And also, this study is deduced some claims and we prove them. We also studied in this paper the relationship between these spaces with a closed cover and the compact space that is characterized by an open cover.

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