Neighborhoods of Multivalent Functions Involving Cho-Kwon-Srivastava Operator

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Abstract

In this paper, we using the Cho-Kwon-Srivastava operator $I_p(\lambda, a, c)$, to define the new subclasses of analytic multivalent functions, certain (n, δ)-neighborhood properties for functions belonging to these classes are obtained.

Keywords: Analytic functions, (n, δ)-neighborhood, Cho-Kwon-Srivastava operator.

الملخص

في هذه الورقة نستخدم مؤثر شو – نو – سرايفستافا $I_p(\lambda, a, c)$، لتعريف فصول جزئية جديدة $H^{p,\lambda}_{n,m}(b, a, c)$ ، $L^{p,\lambda}_{n,m}(b, a, c; \mu)$ من الدوال التحليلية متعددة القيمة، ودراسة خصائص الجوار - (n, δ) للدوال التي تنتمي إلى تلك الفصول الجزئية. كما تم دراسة حدود المعاملات التي تتضمن الشروط الكافية والضرورية لأنتماء الدوال إلى تلك الفصول الجزئية، أيضاً تم دراسة علاقات الإحتواء المتضمنة الجوارات - (n, δ) على تلك الفصول الجزئية. في نهاية هذه الورقة تم دراسة خصائص أخرى للجوار على الفصول المعنية.

1. Introduction

Let $A_p(n)$ be the class of normalized functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k , \qquad (n, p \in N = \{1,2,3, \dots\}). \tag{1.1}$$

Which are analytic and p-valent in the open unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$.

Let $T_p(n)$ be the subclass of $A_p(n)$ consisting functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k , \qquad (a_k \geq 0, n, p \in N), \tag{1.2}$$

which are p-valent in U.

Definition 1. [11]. For $f(z) \in A_p(n), \lambda > -p, a, c \in \mathbb{R} \backslash Z_0^-$, and $p \in N$, the Cho-Kwon-Srivastava operator define as follows:

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$$I_p(\lambda,a,c)f(z)=z^p+\sum_{k=n+p}^{\infty}\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}a_kz^k, \tag{1.3}$$

where

$$(k)_n=\frac{\Gamma(n+k)}{\Gamma(k)}=\begin{cases}k(k+1)\dots(k+n-1), & n\in N\\ 1 & ,\ n=0\end{cases}$$

From (1.3), we deduce that

$$z\left(I_p(\lambda,a,c)f(z)\right)'=(\lambda+p)I_p(\lambda+1,a,c)f(z)-\lambda I_p(\lambda,a,c)f(z)\,. \tag{1.4}$$

We also note that

- **i.** $I_p(0,p,1)f(z)=I_p(1,p+1,1)f(z)=f(z);$
- **ii.** $I_p(1,p,1)f(z)=\frac{zf'(z)}{p}$;
- **iii.** $I_p(n,a,a)f(z)=D^{n+p-1}f(z), n\in N, n>-p.$

The Ruscheweyh derivative of $(n+p-1)$ th order [4].

The operator $I_p(\lambda,a,c)(\lambda>-p,a,c\in\mathbb{R}\backslash Z_0^-)$ was recently introduced by Cho et al. ([1],[11]) who investigated (a mong other things) some inclusion relationships and properties of various subclasses of multivalent functions in A_p,which are defined by main of the operator $I_p(\lambda,a,c)$, for $\lambda=c=1$ and $a=p+n$, the Cho-Kwon-Srivastava operator $I_p(\lambda,a,c)$ yields the Noor integral operator $I_p(1,n+p,1)=I_{n,p}$ $(n>-p)$ of $(n+p-1)$ th order, studied by Liu and Noor [6] (see also [9],[10]). The linear operator $I_1(\lambda,\mu+2,1)(\lambda>-1,\mu>-2)$ was also recently introduced and studied by Choi et. al. [3]. For relevant details about further special cases of the Cho-Saigo-Srivastava operator $I_1(\lambda,\mu+2,1)$ the interested reader may refer to the works by Cho-et. al. [1] and Choi. et. al. [3] (see also [2]).

For any function $f(z)\in T_p(n)$ and $\delta\geq 0$, the (n,δ)-neighborhood of $f(z)$ is defined as,

$$N_{n,\delta}f(z)=\left\{g\in T_p(n): g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k\ and\ \sum_{k=n+p}^{\infty}k|a_k-b_k|\leq\delta\right\}. \tag{1.5}$$

For the function $h(z)=z^p,(p\in N)$, we have

$$N_{n,\delta}(h)=\left\{g\in T_p(n): g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k\ and\ \sum_{k=n+p}^{\infty}k|b_k|\leq\delta\right\}. \tag{1.6}$$

The concept of neighborhoods was first introduced by Goodman [5] and then generalized by Ruscheweyh [12] and Mahzoon [8].

Definition 2. A function $f(z)\in T_p(n)$ is said to be in the class $H_{n,m}^{p,\lambda}(b,a,c)$ if

$$\left|\frac{1}{b}\left(\frac{z\left(I_p(\lambda,a,c)f(z)\right)^{(m+1)}}{\left(I_p(\lambda,a,c)f(z)\right)^{(m)}}-(p-m)\right)\right|<1, \tag{1.7}$$

where $p \in N, m \in N_0, \lambda > -p, a, c \in \mathbb{R}\backslash Z_0^-, p > m, b \in \mathbb{C}\backslash\{0\}$ and $z \in U$.

Definition 3. A function $f(z) \in T_p(n)$ is said to be in the class $L_{n,m}^{p,\lambda}(b,a,c;\mu)$ if;

$$\left|\frac{1}{b}\left[p(1-\mu)\left(\frac{I_p(\lambda,a,c)f(z)}{z}\right)^{(m)}+\mu\left((I_p(\lambda,a,c)f(z))^{(m+1)}\right)-(p-m)\right]\right| < p-m. \tag{1.8}$$

Where $p \in N, m \in N_0, \lambda > -p, a, c \in R\backslash Z_0^-, \mu \geq 0, p > m, b \in \mathbb{C}\backslash\{0\}$ and $z \in U$.

2. Coefficient Bounds.

Theorem 1. Let $f(z) \in T_p(n)$, defined by (1.2), then $f(z)$ is in the class $H_{n,m}^{p,\lambda}(b,a,c)$ if and only if,

$$\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}[k+|b|-p]a_k \leq |b|\binom{p}{m}. \tag{2.1}$$

Proof. Let $f(z) \in H_{n,m}^{p,\lambda}(b,a,c)$. Then by applying to the condition (1.7), we can write,

$$Re\left\{\frac{z(I_p(\lambda,a,c)f(z))^{(m+1)}}{(I_p(\lambda,a,c)f(z))^{(m)}}-(p-m)\right\} > -|b|, \tag{2.2}$$

$$Re\left\{\frac{\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}[p-k]a_k z^{k-p}}{\binom{p}{m}-\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}a_k z^{k-p}}\right\} \geq -|b|. \tag{2.3}$$

Now choose values of z on the real axis and let $z \to 1$ through real values. Then inequalities (2.3), immediately yields the desired condition (2.1).

$$\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}[p-k]a_k \geq -|b|\binom{p}{m}+|b|.\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}a_k,$$

$$\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}[p-k-|b|]a_k$$

$$\geq -|b|\binom{p}{m}.\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}[k+|b|-p]a_k \leq |b|\binom{p}{m}.$$

Conversely, by applying the hypothesis (2.1) and letting $|z|=1$, we obtain,

$$\left|\frac{z\left(I_p(\lambda,a,c)f(z)\right)^{(m+1)}}{\left(I_p(\lambda,a,c)f(z)\right)^{(m)}}-(p-m)\right|$$

$$=\left|\frac{\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}[k-p]a_k z^{k-m}}{\binom{p}{m}z^{p-m}-\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}a_k z^{k-m}}\right|$$

$$\leq \frac{|b|\left[\binom{p}{m}-\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}a_k\right]}{\binom{p}{m}-\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k}{m}a_k}=|b|.$$

Hence by the maximum modulus theorem, we have$f(z)\in H_{n,m}^{p,\lambda}(b,a,c)$. Thus the proof is completed.

On similar lines, we can prove the following theorem.

Theorem 2. Let the function $f(z)\in T_p(n)$ be defined by (1.2), then $f(z)$ is in the class $L_{n,m}^{p,\lambda}(b,a,c;\mu)$ if and only if;

$$\sum_{k=n+p}^{\infty}\left(\frac{(c)_{k-p}(\lambda+p)_{k-p}}{(a)_{k-p}(1)_{k-p}}\right)\binom{k-1}{m}[p-p\mu+\mu k]a_k$$

$$\leq (p-m)\left[\frac{|b|-1}{m!}+\binom{p}{m}\right]. \tag{2.4}$$

3. Inclusion Relationships Involving (n,δ)-Neighborhoods

Theorem 3. If;

$$\delta=\frac{(n+p)|b|\binom{p}{m}}{(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}}\,,\qquad (p>|b|). \tag{3.1}$$

Then $H_{n,m}^{p,\lambda}(b,a,c)\subset N_{n,\delta}(h)$.

Proof. Let $f(z)\in H_{n,m}^{p,\lambda}(b,a,c)$, by Theorem 1, we have

$$(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k|b|\binom{p}{m}$$

which implies,

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|\binom{p}{m}}{(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}}. \tag{3.2}$$

Using (2.1) and (3,2), we have,

$$\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}\sum_{k=n+p}^{\infty} ka_k$$

$$\leq |b|\binom{p}{m}+(p-|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k$$

$$\leq |b|\binom{p}{m}+(p-|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}\frac{|b|\binom{p}{m}}{(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}}$$

$$= |b|\binom{p}{m}\frac{n+p}{n+|b|}.$$

That is,

$$\sum_{k=n+p}^{\infty} ka_k \leq \frac{|b|(n+p)\binom{p}{m}}{(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}} = \delta, \qquad (p>|b|).$$

Thus, by the Definition given by (1.6), $f(z) \in N_{n,\delta}(h)$. This completes the proof.

Similarly, we prove the following theorem.

Theorem 4. If;

$$\delta = \frac{(p-m)(n+p)\left[\frac{|b|-1}{m!}+\binom{p}{m}\right]}{[\mu(n+p-1)+1]\left[\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p-1}{m}\right]}, \qquad (\mu>1) \tag{3.3}$$

Then $L_{n,m}^{p,\lambda}(b,a,c;\mu) \subset N_{n,\delta}(h)$.

4. Further Neighborhood Properties

In this section, we determine the neighborhood properties for each the classes $H_{n,m}^{p,\lambda,\alpha}(b,a,c)$ and $L_{n,m}^{p,\lambda,\alpha}(b,a,c;\mu)$ which we define as follows:

For $0 < \alpha < p$ and $z \in U$, a function $f(z)$ is said to be in the class $H_{n,m}^{p,\lambda,\alpha}(b,a,c)$ if there exists a function $g(z) \in H_{n,m}^{p,\lambda}(b,a,c)$ such that;

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha. \tag{4.1}$$

Analogously for $0 < \alpha < p$ and $z \in U$, a function $f(z)$ is said to be in the class $L_{n,m}^{p,\lambda,\alpha}(b,a,c;\mu)$. If there exists a function $g(z) \in L_{n,m}^{p,\lambda}(b,a,c;\mu)$ such that the inequality (4.1) holds true.

Theorem 5. If $g(z) \in H_{n,m}^{p,\lambda}(b,a,c)$ and;

$$\alpha = p - \frac{\delta(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}}{(n+p)\left[(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m} - |b|\binom{p}{m}\right]}\ , \tag{4.2}$$

then $N_{n,\delta}(g) \subset H_{n,m}^{p,\lambda,\alpha}(b,a,c)$.

Proof. Let $f(z) \in N_{n,\delta}(g)$, Then we find from (1.5), that;

$$\sum_{k=n+p}^{\infty} |a_k - b_k| \leq \delta, \tag{4.3}$$

which yields the coefficient inequality,

$$\sum_{k=n+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+p}; (n \in N), \tag{4.4}$$

Since $g(z) \in H_{n,m}^{p,\lambda}(b,a,c)$ by (3.2), we have;

$$\sum_{k=n+p}^{\infty} b_k \leq \frac{|b|\binom{p}{m}}{(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m}}\ , \tag{4.5}$$

so that,

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=n+p}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+p}^{\infty} b_k}$$

$$\leq \frac{\delta}{n+p}\frac{(n+|b|)\binom{n+p}{m}}{\left[(n+|b|)\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p}{m} - |b|\binom{p}{m}\right]}.$$

$$= p - \alpha\ ,$$

Thus, by Definition, $f(z) \in H_{n,m}^{p,\lambda,\alpha}(b,a,c)$ for α is given by (4.1).Thus the proof is completed.

The proof of Theorem 6 below is similar to that of Theorem 5 above, therefore, we omit the details involved.

Theorem 6. If $g(z) \in L_{n,m}^{p,\lambda}(b,a,c;\mu)$ and;

$$\alpha = p - \frac{\delta[\mu(n+p-1)+1]\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p-1}{m}}{(n+p)\left[[\mu(n+p-1)+1]\left(\frac{(c)_n(\lambda+p)_n}{(a)_n(1)_n}\right)\binom{n+p-1}{m} - (p-m)\left(\frac{|b|-1}{m!}+\binom{p}{m}\right)\right]}, \tag{4.6}$$

then $N_{n,\delta}(g) \subset L_{n,m}^{p,\lambda}(b,a,c;\mu)$.

Remark 1. In the special case when $\lambda = n \in N, a = c$, the result due to R. K. Raina and H. M. Srivastava [13].

Remark 2. In the special case when $m = 0, p = 1, b = \gamma\beta, (0 < \beta \leq 1; \gamma \in \mathbb{C}\{0\})$ and $\lambda = n, a = c$, the result due to S. Latha and L. Dileep [7].

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