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ترسل البحوث وجميع المراسلات المتعلقة بالمجلة إلى العنوان التالي:

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شروط وتعليمات النشر

1- أن يكون البحث أصيلاً ومبتكراً ولم يسبق نشره في أي جهة أخرى، وتتوفر فيه شروط البحث العلمي المعتمدة على الأصول العلمية والمنهجية المتعارف عليها في كتابة البحوث الاكاديمية.
2- أن يكون البحث مكتوباً بلغة سليمة، ومراعياً لقواعد الضبط ودقة الرسوم والاشكال – إن وجدت – ومطبوعاً بنط (14) وبخط (Simplified Arabic)، وألا تزيد صفحات البحث عن (35) صفحة متضمنة الهوامش والمراجع.
3- يجب أن يشتمل البحث على العناصر التالية: - عنوان البحث باللغتين العربية والإنجليزية؛ - ملخص تنفيذي باللغتين العربية والإنجليزية في نحو 100-125 كلمة، والكلمات المفتاحية (keywords) بعد الملخص.
4- يتم توثيق الهوامش وفق طريقة **APA** (طريقة الجمعية الأمريكية السيكولوجية**)** بإصدارتها المختلفة.
5- يُفضل أن تكون الجداول والاشكال مدرجة في أماكنها الصحيحة، وأن تشمل العناوين والبيانات الإيضاحية الضرورية، ويراعى ألا تتجاوز أبعاد الاشكال والجداول حجم حيز الكتابة في صفحة Microsoft Word.
6- أن يكون البحث ملتزماً بدقة التوثيق، وحسن استخدام المصادر والمراجع، وأن تثبت مصادر ومراجع البحث في نهاية البحث.
7- تحتفظ المجلة بحقها في اخراج البحث وإبراز عناوينه بما يتناسب واسلوبها في النشر.
8- ترحب المجلة بنشر البحوث المكتوبة باللغة الأجنبية ويفضل أن يرفق البحث بملخص باللغة العربية (لا يتجاوز 200 كلمة).
9- ترحب المجلة بنشر ما يصلها من ملخصات الرسائل الجامعية التي تمت مناقشتها وإجازتها، على أن يكون الملخص من إعداد صاحب الرسالة نفسه.
10- تُرسل نسخة من البحث مطبوعة على ورق بحجم (A4) إلى مقر المجلة، ونسخة إلكترونية إلى إيميل المجلة: rwafedalmarefa@gmail.com، على أن يدون على صفحة الغلاف: اسم الباحث، لقبه العلمي، مكان عمله، تخصصه، رقم هاتفه وبريده الإلكتروني.
11- يخطر الباحث بقرار صلاحية بحثه للنشر من عدمها خلال مدة ثلاثة أشهر من تاريخ استلام البحث.
12- في حالة ورود ملاحظات وتعديلات على البحث من المحكم، ترسل تلك الملاحظات إلى الباحث لإجراء التعديلات اللازمة بموجبها، على أن تعاد للمجلة خلال مدة أقصاها شهر واحد.
13- الأبحاث التي لم تتم الموافقة على نشرها لا تعاد إلى الباحثين.
14- تؤول جميع حقوق النشر للمجلة.
15- دفع رسوم التحكيم العلمي والمراجعة اللغوية والنشر، إن وجدت.

البحوث المنشورة في هذه المجلة تعبر عن رأي أصحابها ولا تعبر بالضرورة عن رأي المجلة أو الجامعة.

الكلمة الافتتاحية

بسم الله الرحمن الرحيم، عليه نتوكل وبه نستعين، نحمده سبحانه كما ينبغي أن يُحمد، ونصلي ونسلم على رسوله محمد وعلى آله وصحبه والتابعين.

وبعـــد،،،

إن سبيل نهضة الأمم إنما يكون بالبحث العلمي في شتى المجالات، فدوره مهم لمواكبة التقدم والرقي بالمجتمع فبالبحث العلمي ينمى القدرات البشرية وهو الأساس في الابتكاروالإبداع. بعون من الله وتوفيقه، وبعد الجهد الكبير الذي بذلته هيئة التّحرير تكاملت الاستعدادات لإصدار العدد التاسع من مجلة رو افد المعرفة، والذي نأمل أن يلبي طموحات المهتمين والباحثين. ومن هنا ندعو كل الباحثين والكتاب الإسهام في استمرار المجلة بتقديم نتاجهم العلمي للنشر، ونرحب بآراء القراء والباحثين ونقدهم البناء حتى تخرج المجلة في صورتها المثلى وليكون العدد التالي أفضل من سابقه. وختاما يجدر بنا مع إصدار هذا العدد والذي يحتوي على عدد أربعة عشر بحثاً أصيلاً مختلفاً، أن نتقدم بجزيل الشكر والتقدير للمحكمين والمؤلفين وكل من أسهم في إخراجه وتصميمه، آملين أن تكون محتوياته نافعة للجميع.

والحمد لله في بدءٍ ومُخْتَتَمِ.

هيئة التحرير

.

المحتويات

Generalized Systems of Impulsive Differential Equations

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Abstract:

With regard to Impulsive differential equations (IDE) and their applications, this paper shows Gronwall Inequality Bellman Lemma for Impulsive differential equations. Another point is shown here in this paper is necessary condition for periodic solution of (IDE). This paper also solves and extends linear homogenous impulsive differential systems with their stability respectively.

Keywords: Gronwall Inequality Bellman Lemma, Periodic systems, Linear system of Impulsive DE, Stability.

الملخص:

تدرس هذا الورقة مفاهيم المعادلات تفاضيلة الإندفاعية وتطبيق نظرية المساعدة للمتباينة جرونوال في المعادلات التفاضلية الإندفاعية. هناك نقطة أخرى موضحة هنا في هذه الورقة وهي شرط ضروري للحل الدوري للمعادلات التفاضلية الإندفاعية. هذا البحث يحل ويوسع منظومة المعادلات التفاضلية الإندفاعية الخطية المتجانسة مع إثباتها على التوالي.

الكلمات المفتاحية: نظرية المساعدة للمتباينة جرونوال، النظام الدوري، النظام الخطي، الاستقرار.

1. Introduction

The necessity to study impulsive differential equations (IDE) is due to the fact that these equations are useful mathematical tools in modelling of many real processes and phenomena studied in optimal control, mechanics, biology, electronic, economics, medicine, etc.

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of the state abruptly.

The differential equations that are involving impulse effects is called IDE, which appear as natural description of observed evolution phenomena of several real-world problems. Such as: for blood, muscle, tissue, etc. and finally is eliminated from the system by the kidneys see for more details ([1-5]. It is easily to release that, there are many good monographs on the impulsive differential equations [6-9].

Nevertheless, there is a different situation in many physical phenomena that have a rapid change in their situations for example mechanical systems with impact, biological systems heart beats for instance, blood flows, population dynamics [10-14], theoretical physics, radio physics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry [15], engineering [16], control theory [17,18] and so on. The first paper in this theory is related to [19]. Although this theory has undergone several developments, it is considered slow due to the special features of impulsive differential equations (pulse phenomena for instance).

In many previous works, first and second order ordinary differential equations were treated with impulses (see [1, 8, 11, 15-23]). An IDE is described by three components:

1. a continuous-time differential equation, which governs the state of the system between impulses;
2. an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs;
3. and a jump criterion, which defines a set of jump events in which the impulse equation is active.

The paper is organized as follows: The system of IDE has been discussed in the next section. The proof both Gronwall-Bellman Lemma for DE and IDE are shown in Section 3. In Section 4, the continuous dependences and periodic systems are discussed. Section 5 is devoted to Linear and stability of IDE. In Section 6, conclusions are drawn.

2. The idea of IDE Description

Let us consider the following system

$$\left.\begin{array}{l} x' = f(t,x), t \neq \theta_i \\ \Delta x|_{t=\theta_i} = I_i(x) \end{array}\right\}; \ \forall\, (t,x) \in R \qquad (2.1)$$

Here $\Delta x|_{t=\theta_i}$ denotes the jump at $t = \theta_i$ and we have $\Delta x|_{t=\theta_i} = x(\theta_i^+) - x(\theta_i^-)$, where

$$x(\theta_i^+) = \lim_{t\to\theta_i^+} x(t) \ \&\ x(\theta_i^-) = \lim_{t\to\theta_i^-} x(t)$$

Remark 2.1:

We assume that the solutions of $x' = f(t,x)$ exists on the intervals $(\theta_i, \theta_{i+1}]$ that is, the solution can be continued to the jump points

Example 2.1:

If $x' = 0,\ t \neq i\ \&\ \Delta x|_{t=i} = -x + \frac{1}{x},\ t_0 \in (0,1), x_0 = a$ and $a \neq 0$.

To solve for $t \in [t_0, 1]$, we have $x' = 0\ \&\ x(t_0) = a$, then, we obtain $x(t) = a$, and we need $x(1^+)$, By using impulse condition

$$\Delta x|_{t=1} = -x(1) + \frac{1}{x(1)} = x(1^+) - x(1^-) \Rightarrow x(1^+) = \frac{1}{x(1)} = \frac{1}{a}$$

For $t \in (1,2]$, we have $x' = 0$ & $x(1^+) = \frac{1}{a}$, then, we obtain $x(t) = \frac{1}{a}$, and we need $x(2^+)$, by using impulse condition

$$\Delta x|_{t=2} = -x(2) + \frac{1}{x(2)} = x(2^+) - x(2^-) \Rightarrow x(2^+) = \frac{1}{x(2)} = \mathrm{a}$$

For $\mathrm{t} \in (2,3]$, we have $\mathrm{x}' = 0$ & $\mathrm{x}(2^+) = \mathrm{a}$, then, we obtain $\mathrm{x(t)} = \mathrm{a}$ and so on for $\mathrm{t} > \mathrm{t}_0$, the solution is shown in the Figure 1.

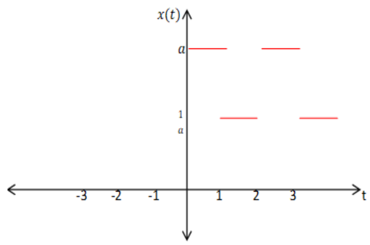


Figure 1: The numerical solution of IDE for $\mathrm{t} > \mathrm{t}_0$.

What about $x(t)$ for $t < t_0$?

For $t \in (0, t_0]$, we have $x' = 0$ & $x(t_0) = a$, then, we obtain, $x(t) = a$, and we need $x(0^-)$, we need to move backward

$$\Delta x|_{t=0} = -x(0) + \frac{1}{x(0)} = x(0^+) - x(0^-) \Rightarrow x(0) = \frac{1}{x(0^+)} = \frac{1}{a}$$

This is for $t \in (-1,0]$, we have $x' = 0$ & $x(-1^+) = \frac{1}{a}$, then, we obtain $x(t) = \frac{1}{a}$, and we need $x(-1)$. By using impulse condition

$$\Delta x|_{t=-1} = -x(-1) + \frac{1}{x(-1)} = x(-1^+) - x(-1^-) \Rightarrow$$

$$x(-1) = \frac{1}{x(-1^+)} = a$$

This is for $t \in (2,3]$, we have $x' = 0$ & $x(-1) = a$, and then, we obtain $x(t) = a$ and so on for $t < t_0$

General solution $x(t, t_0, a) = \begin{Bmatrix} \frac{1}{a}, \ if\ 2n-1 < t \leq 2n \\ a, \ if\ 2n < t \leq 2n+1 \end{Bmatrix}$, the general solution is shown in the figure 2:

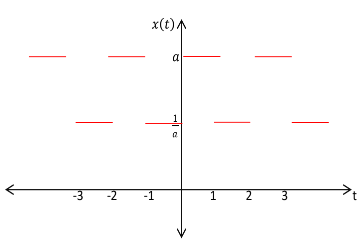


Figure 2: The general numerical solution of IDE for.

What about $x(t, t_0, 0)$, for $t \in [t_0, 1]$,

$$\left.\begin{array}{l} x' = 0 \\ x(t_0) = 0 \end{array}\right\} \Rightarrow \text{solution is } x(t) = 0.$$

We need $x(1^+)$?

$$\Delta x|_{t=1} = -x(1) + \frac{1}{x(1)} = x(1^+) - x(1^{-1}) \Rightarrow x(1^+) = \frac{1}{x(1)}$$

Since $x(1)$=0, $x(1^+)$ is not defined, that is, the solution $x(t, t_0, 0)$ is not defined for $t > 1$ in fact, it is defined only on $(0, t_0]$,

What about the stability of the solution $x(t, t_0, a)$?, it is important to recall:

$$\left.\begin{array}{l} x' = f(t,x) \\ x(t_0) = x_0 \end{array}\right\} \Rightarrow \text{solution is } x(t, t_0, x_0)$$

The solution $x(t, t_0, x_0)$ is stable if $\forall \varepsilon > 0,\ \exists \delta > 0$ such that, for all $\mathrm{t} \geq \mathrm{t}_0$

$$|x_0 - x_1| < \delta \Rightarrow |x(t, t_0, x_0) - x(t, t_0, x_1)| < \varepsilon,\ \forall\ t \geq t_0.$$ [11] and [15].

Let us go back to our solution of the impulsive differential equations in Example 1.

$$x(t, t_0, a) = \begin{cases} \frac{1}{a}, & if\ 2n-1 < t \leq 2n \\ a, & if\ 2n < t \leq 2n+1 \end{cases}$$

To discuss the stability of $x(t, t_0, a)$ is needed to consider the following cases:

Case 1: when $a > 0$

$$|a - b| < \delta \Rightarrow |x(t, t_0, a) - x(t, t_0, b)| < \varepsilon,\ \forall\ t \geq t_0$$

To check this condition, exit or not if and only if

$$t \in (2n, 2n+1] \Rightarrow |x(t, t_0, a) - x(t, t_0, b)| = |a - b|$$

Where, if

$$t \in (2n-1, 2n] \Rightarrow |x(t, t_0, a) - x(t, t_0, b)| = \left|\frac{1}{a} - \frac{1}{b}\right|$$

Assume that:

$$\delta \leq \frac{a}{2} \text{ (so that } b > 0) \text{ then } b \in \left(\frac{a}{2}, \frac{3a}{2}\right)$$

$$|x(t, t_0, a) - x(t, t_0, b)| \leq \left|\frac{\delta}{ab}\right| = \frac{\delta}{ab} < \frac{\delta}{a\frac{a}{2}} = \frac{2\delta}{a^2} \leq \varepsilon.$$

The last number is $\leq \varepsilon$ if $\delta \leq \frac{\varepsilon a^2}{2}$ in conclusion given ε>0 if we choose δ=min$\{\varepsilon, \frac{a}{2}, \frac{\varepsilon a^2}{2}\}$ then $|a - b| < \delta$ implies $|x(t, t_0, a) - x(t, t_0, b)| < \varepsilon,\ \forall\ t \geq t_0$. That is if a>0 then the solution $x(t, t_0, a)$ in this case is stable

Case 2: - when if $a < 0$, we know the definition of the stability so, if $t \in (2n, 2n+1]$, then $\forall \mathcal{E} > 0,\ \exists \sigma > 0$ such that

$$|a - b| \leq \sigma \Longrightarrow |x(t, t_0, a) - x(t, t_0, b)| \leq \mathcal{E}$$

Whereas if $t \in (2n-1, 2n]$, then

$$|a - b| \leq \sigma \Longrightarrow |x(t, t_0, a) - x(t, t_0, b)| = \left|\frac{1}{a} - \frac{1}{b}\right| = \frac{|-(a-b)|}{|ab|} \leq \frac{|a-b|}{ab} \leq \frac{\sigma}{ab}$$

$$|a - b| \leq \sigma \Longrightarrow -\sigma \leq a - b \leq \sigma \Longrightarrow -\sigma + a \leq b \leq \sigma + a.$$

Let us assume that $\sigma \leq -\frac{a}{2}$, and $b < 0$, hence $b \in [\frac{3a}{2}, \frac{a}{2}]$. Then, we get

$$|x(t,t_0,a)-x(t,t_0,b)| = \left|\frac{1}{a}-\frac{1}{b}\right| = \frac{|-(a-b)|}{|ab|} \le \frac{|a-b|}{ab} \le \frac{\sigma}{ab} \le \frac{\sigma}{a\frac{a}{2}} = \frac{2\sigma}{a^2} \le \mathcal{E}.$$

So, $\sigma \le \frac{a^2\mathcal{E}}{2}$. In conclusion for all $\mathcal{E} > 0$ if we choose $\sigma = \min\{\mathcal{E}, -\frac{a}{2}, \frac{a^2\mathcal{E}}{2}\}$, then $|a-b| \le \sigma$ implies $|x(t,t_0,a)-x(t,t_0,b)| \le \mathcal{E}$ for all $t \ge t_0$ that is if < 0 then the solution $x(t,t_0,a)$ is stable and the figure below shows our solution is periodic function.

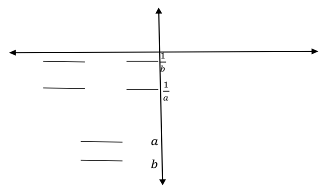


Figure 3: The solution of IDE if $a < 0$.

Case3: - if $a = 0$, then our solution for impulse exists only between (0,1), but in general case the solution is not defined. Let us go back to our equation it has zero solution when $a = 0$, for the stability in this case, it is not stable, it is bifurcation solution when a=0. because there is no any equilibrium point at 0.

Definition 2.1:[8]

Suppose that $I \subset R$ and $\{\theta_i\} \subset I$. A function φ is said to be in piecewise continuous on the interval (I) ($PC(I)$) if φ is left continuous on I and φ has jump discontinuity at $t = \theta_i$ for all $i = 1,2,\ldots$

Lemma 2.1: if $\varphi' \in PC(I)$ then

$$\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} \varphi'(s)ds + \sum_{t_0 \le \theta_i < t} \Delta\, \varphi(\theta_i).$$

Note that for continuous $\Delta\, \varphi(\theta_i)$ must $\varphi(\theta_i^+) - \varphi(\theta_i^-) = 0$

Proof:

Let

$$\pi(t) = \sum_{t_0 \le \theta_i < t} \Delta\, \varphi(\theta_i)$$

and

$$h(t) = \varphi(t_0) + \int_{t_0}^{t} \varphi'(s)ds + \pi(t).$$

Note that for $t \in [t_0, \theta_1]$, we have $\pi(t)=0$, for $t \in (\theta_1, \theta_2]$, We have $\pi(t) = \Delta\varphi(\theta_1)$ for $t \in (\theta_2, \theta_3]$, We have $\pi(t) = \Delta\varphi(\theta_1) + \Delta\varphi(\theta_2)$, for $t_0 \le \theta_i < t$ as shown in the Figure 4.

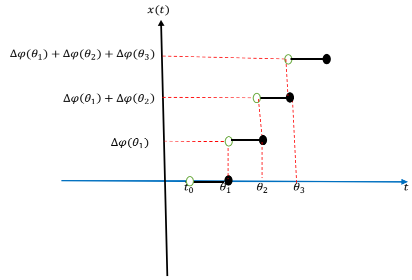


Figure 4: The steps when $t_0 \leq \theta_i < t$.

This means that if $t \neq \theta_i$, then $\pi'(t) = 0$ and hence if $t \neq \theta_i$, we have

$$h'(t) = \frac{d}{dt}\left[\varphi(t_0) + \int_{t_0}^{t} \varphi'(s)ds + \pi(t)\right] = \varphi'(t)$$

Moreover,

$$\Delta h(\theta_i) = \Delta h(\theta_i^+) - \Delta h(\theta_i^-) = \varphi(t_0) + \int_{t_0}^{\theta_i^+} \varphi'^{(s)}ds + \pi(\theta_i^+) - \left[\varphi(t_0) + \int_{t_0}^{\theta_i^-} \varphi'(s)ds + \pi(\theta_i^-)\right]$$

$$= \pi(\theta_i^+) - \pi(\theta_i^-) = \sum_{\substack{t_0 \leq \theta_k < \theta_i^+ \\ k=1,2,3,\dots,i}} \Delta\ \varphi(\theta_k) - \sum_{\substack{t_0 \leq \theta_k < \theta_i^- \\ k=1,2,3,\dots,i-1}} \Delta\ \varphi(\theta_k) = \Delta\varphi(i)$$

That is

$$h'(t) = \varphi'(t), t \neq \theta_i$$

$$\Delta h(\theta_i) = \Delta\varphi(i) \text{ and } h(t_0) = \varphi(t_0).$$

And hence

$$h(t) = \varphi(t)$$

More precisely if, $\varphi(t) \in PC'(I)$, then

$$\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} \varphi'(s\,)ds + \sum_{t_0 \leq \theta_i < t} \Delta\ \varphi(\theta_i)$$

3. Ronwall-Bellman Lemma for DE.

Lemma 3.1:[9]

Let C≥ 0, v be positive continuous function, $u \in C'(I)$ and positive such that

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} u(s)v(s)ds \;\text{ for all t} \geq t_0$$

Then

$$u(t) \leq \mathrm{C} + \mathrm{e}^{\int_{t_0}^{t} v(s)ds}$$

Lemma 3.2: (Gronwall-Bellman Lemma for IDE)

Let C$\geq 0, \mathrm{B}_i \geq 0$and $v \in PC'(I)$, $v > 0$ continuous function and $u \in PC'(I)$is positive and satisfies

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} u(s)v(s)ds + \sum_{t_0 \leq \theta_i < t} \beta_i u(\theta_i) \qquad (3.1)$$

for all t $\geq t_0$

Then

$$u(t) \leq ce^{\int_{t_0}^{t} v(s)ds} \prod_{t_0 \leq \theta_i < t} (1 + \beta_i) \qquad (3.2)$$

Proof:

We will use induction on the interval $(\theta_i, \theta_{i+1}]$, then when $t \in [t_0, \theta_1]$, hence the inequality (3.1) becomes

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} u(s)v(s)ds \qquad (3.3)$$

By using G.B Lemma, we have

$$u(t) \leq C\mathrm{e}^{\int_{t_0}^{t} v(s)ds} \quad \forall t \in [t_0, \theta_1] \qquad (3.4)$$

Which means that (3.2) satisfies for $t \in [t_0, \theta_1]$ by (3.4). Let $t \in (\theta_1, \theta_2]$, then the inequality (2.1) becomes

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} u(s)v(s)ds + B_1 u(\theta_1) \qquad (3.5)$$

$$= \mathrm{C} + \int_{t_0}^{\theta_1} u(s)v(s)ds + \int_{\theta_1}^{t} u(s)v(s)ds + B_1 u(\theta_1)$$

By using the Equation (3.4)

$$\leq \mathrm{C} + \int_{t_0}^{\theta_1} u(s)v(s)ds + \int_{\theta_1}^{t} u(s)v(s)ds + B_1 C\mathrm{e}^{\int_{t_0}^{\theta_1} v(s)ds}$$

By using G.B Lemma 3.2, we have

$$\leq Ce^{\int_{t_0}^{\theta_1} v(s)ds} + \int_{\theta_1}^{t} u(s)v(s)ds + B_1Ce^{\int_{t_0}^{\theta_1} v(s)ds} = \mathrm{C}(1+\mathrm{B}_1)e^{\int_{t_0}^{\theta_1} v(s)ds} + \int_{\theta_1}^{t} u(s)v(s)ds$$

By using G.B Lemma, we have

$$u(t) \leq C(1+B_1)e^{\int_{t_0}^{\theta_1} v(s)ds}e^{\int_{\theta_1}^{t} v(s)ds} \Longrightarrow$$

$$u(t) \leq C(1+B_1)\, e^{\int_{t_0}^{t} v(s)ds} \quad \forall t \in (\theta_1, \theta_2] \qquad (3.6)$$

Which means that (2.2) satisfies for $t \in (\theta_1, \theta_2]$ by (3.6). Let $t \in (\theta_2, \theta_3]$, then the inequality (1) becomes as (3.7)

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} u(s)v(s)ds + B_1u(\theta_1) + B_2u(\theta_2) \qquad (3.7)$$

$$= \mathrm{C} + \int_{t_0}^{\theta_2} u(s)v(s)ds + B_1u(\theta_1) + \int_{\theta_2}^{t} u(s)v(s)ds + B_2u(\theta_2)$$

By using the Equation (3.6)

$$u(t) \leq C(1+B_1)e^{\int_{t_0}^{\theta_2} v(s)ds} + \int_{\theta_2}^{t} u(s)v(s)ds + B_2C(1+B_1)\, e^{\int_{t_0}^{\theta_2} v(s)ds}$$

$$= C(1+B_1)(1+B_2)e^{\int_{t_0}^{\theta_2} v(s)ds} + \int_{\theta_2}^{t} u(s)v(s)ds$$

By using G.B Lemma 3.2, we have

$$u(t) \leq C(1+B_1)(1+B_2)e^{\int_{t_0}^{\theta_2} v(s)ds}e^{\int_{\theta_2}^{t} v(s)ds} \Longrightarrow$$

$$u(t) \leq C(1+B_1)(1+B_2)e^{\int_{t_0}^{t} v(s)ds} \quad \forall t \in (\theta_2, \theta_3] \quad (3.8)$$

The inequality (3.8) implies (3.2), Assume that the inequality (3.1) is true for $t \in (\theta_k, \theta_{k+1}]$. That means

$$u(t) \leq \mathrm{C} + B_1u(\theta_1) + B_2u(\theta_2) + \cdots + B_ku(\theta_k) + \int_{t_0}^{t} u(s)v(s)ds \qquad (3.9)$$

implies

$$u(t) \leq ce^{\int_{t_0}^{t} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k}} (1+\beta_i) \qquad (3.10)$$

Now, we need to prove that the inequality (3.1) satisfies (3.2) when $t \in (\theta_{k+1}, \theta_{k+2}]$. Then the inequality (3.1) becomes

$$u(t) \leq C + B_1 u(\theta_1) + B_2 u(\theta_2) + \cdots + B_k u(\theta_k) + B_{k+1} u(\theta_{k+1}) + \int_{t_0}^{t} u(s)v(s)ds \quad (3.11)$$

$$= C + B_1 u(\theta_1) + B_2 u(\theta_2) + \cdots + B_k u(\theta_k) + \int_{t_0}^{\theta_{k+1}} u(s)v(s)ds + B_{k+1} u(\theta_{k+1})$$

$$+ \int_{\theta_{k+1}}^{t} u(s)v(s)ds$$

By using the Equation (3.10), then (3.11) becomes

$$u(t) \leq c e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k}} (1+\beta_i) + B_{k+1} c e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k}} (1+\beta_i) + \int_{\theta_{k+1}}^{t} u(s)v(s)ds$$

$$= c e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k+1}} (1+\beta_i) + \int_{\theta_{k+1}}^{t} u(s)v(s)ds$$

By using G.B Lemma 3.2, we have

$$u(t) \leq C e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} e^{\int_{\theta_{k+1}}^{t} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k+1}} (1+\beta_i) \Longrightarrow$$

$$u(t) \leq C e^{\int_{t_0}^{t} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k+1}} (1+\beta_i) \ \forall t \in (\theta_{k+1}, \theta_{k+2}] \quad (3.12)$$

The inequality (3.12) implies (3.2). Then the inequality (3.1) implies (3.2) for all $\forall t \in (\theta_{i+1}, \theta_{i+1}]$

Corollary 3.1: If $u \in PC(I)$ is a nonnegative function and C, ζ, L are nonnegative constants such that

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} [\zeta + Lu(s)]ds + \sum_{t_0 \leq \theta_i < t} [\zeta + Lu(\theta_i)] \quad (*)$$

Then,

$$u(t) \leq \left(C + \frac{\zeta}{L}\right) e^{L(t-t_0)} (1+L)^{i(t_0,t)} - \frac{\zeta}{L} \quad (**)$$

Where $i(t,t_0)$ is the number of θ_i in (t,t_0)

Proof:

Let $\varphi(t)=\zeta+Lu(t)\Longrightarrow u(t)=\frac{\varphi(t)-\zeta}{L}$

$$u(t)\le \mathrm{C}+\int_{t_0}^{t}\varphi(s)ds+\sum_{t_0\le\theta_i<t}\varphi(\theta_i)\Longrightarrow$$

$$\zeta+Lu(t)=\varphi(t)\le\zeta+L\left[\mathrm{C}+\int_{t_0}^{t}\varphi(s)ds+\sum_{t_0\le\theta_i<t}\varphi(\theta_i)\right]$$

$$=\zeta+LC+\int_{t_0}^{t}L\varphi(s)ds+\sum_{t_0\le\theta_i<t}L\varphi(\theta_i)$$

By using G.B Lemma 3.2 for IDE, we will have

$$\varphi(t)\le(\zeta+LC)e^{\int_{t_0}^{t}Lds}\prod_{\substack{t_0\le k<t\\ k=1,2,\dots,i}}(1+L)=(LC+\zeta)e^{L(t-t_0)}(1+L)^{i(t_0,t)}$$

Since $u(t)=\frac{\varphi(t)-\zeta}{L}$, then

$$\frac{\varphi(t)-\zeta}{L}\le(\frac{\zeta}{L}+C)e^{L(t-t_0)}(1+L)^{i(t_0,t)}-\frac{\zeta}{L}\Longrightarrow u(t)$$
$$\le\left(C+\frac{\zeta}{L}\right)e^{L(t-t_0)}(1+L)^{i(t_0,t)}-\frac{\zeta}{L}$$

Where $i(t,t_0)$ is the number of θ_i in (t,t_0)

4. Continuous DEPENDENCES AND Periodic Systems

Definition 4.1[15]: Consider

$$\left.\begin{array}{l} x'=f(t,x),t\ne\theta_i\\ \Delta x|_{t=\theta_i}=I_i(x)\end{array}\right\}\qquad(4.1)$$

Where $\theta_i\longrightarrow\infty$ as $i\longrightarrow\infty$, the solution $x(t,t_0,x_0)$ is said to be continuously depending on the initial value, on the interval $[t_0,t_0+T]$ if for all $\varepsilon>0$ there exists $\delta>0$ such that, $|x_0-x_1|<\delta$ implies $|x(t,t_0,x_0)-x(t,t_0,x_1)|<\varepsilon,\ \forall[t_0,t_0+T]$.

Definition 4.2 [15]**:** Consider (4.1) and

$$\left.\begin{array}{l} y'=f(t,y)+g(t,y),t\ne\theta_i\\ \Delta y|_{t=\theta_i}=I_i(y)+W_i(y)\end{array}\right\}\qquad(4.2)$$

The solution $x(t,t_0,x_0)$ of (4.1) is said to be continuously depending on the right side on the interval $[t_0,t_0+T]$ if for all $\varepsilon>0$ there exists $\delta>0$ such that $|g(t,y)|<\delta,|W_i(y)|<\delta$

implies $|x(t,t_0,x_0)-y(t,t_0,x_0)|<\varepsilon,\ \forall t\in[t_0,t_0+T]$ where $y(t,t_0,x_0)$ is the solution of (6) satisfies $y(t_0)=x_0$.

Theorem (4.1):

If $|f(t,x)-g(t,y)|+|I_i(x)-I_i(y)|\le L|x-y|$ for all $i=1,2,\dots P, L>0$, then the solution of the equation (4.1) continuously depends on the initial value and on the right side on interval $[t_0,t_0+T]$

Proof

Let ε>0 be given chosen $\delta>0$ such that $0<\delta<\varepsilon e^{-LT}(1+L)^{-P}$

Now

$$x_0(t)=x_0+\int_{t_0}^{t} f(s\,,x_0(s))ds+\sum_{t_0\le\theta_i<t} I_i(x_0(\theta_i))$$

$$x_1(t)=x_1+\int_{t_0}^{t} f(s\,,x_1(s))ds+\sum_{t_0\le\theta_i<t} I_i(x_1(\theta_i))$$

If $|x_0-x_1|\le\delta$, we have

$$\begin{aligned}|x_0(t)-x_1(t)|&\\ &\le|x_0-x_1|\\ &+\int_{t_0}^{t}|f(s\,,x_0(s))-f(s\,,x_1(s))|ds+\sum_{t_0\le\theta_i<t}\left|I_i\big(x_0(\theta_i)\big)-I_i\big(x_1(\theta_i)\big)\right|\\ &\le\delta+\int_{t_0}^{t}L|x_0(s)-x_1(s)|ds+\sum_{t_0\le\theta_i<t}L|x_0(\theta_i)-x_1(\theta_i)|\end{aligned}$$

By using G.B lemma 3.2 for IDE, we have

$$|x_0(t)-x_1(t)|\le\delta e^{L(t-t_0)}(1+L)^{i(t_0,t)}\le\delta e^{LT}(1+L)^P<\varepsilon$$

Therefore, the solution $x_0(t)$ continuously depends on x_0. To prove the continuous dependence on the right side. Let $\boldsymbol{x}(t)=x(t,t_0,x_0)$ be solution of (5) and $y(t)=y(t,t_0,x_0)$ be solution of (4.2). Let ε>0 be given chosen $\delta>0$ such that $0<\delta<L\varepsilon[e^{LT}(1+L)^P-1]^{-1}$, now

$$x(t)=x_0+\int_{t_0}^{t} f(s\,,x(s))ds+\sum_{t_0\le\theta_i<t} I_i(x(\theta_i))$$

$$y(t)=x_0+\int_{t_0}^{t}\big[f\big(s\,,y(s)\big)+g(s\,,y(s))\big]ds+\sum_{t_0\le\theta_i<t}\big[I_i\big(y(\theta_i)\big)+W_i(y(\theta_i))\big]$$

If $|g(t,y)|<\delta, |W_i(y)|<\delta$, then, we have

$$|x(t)-y(t)| \le \int_{t_0}^{t}[L|x(s)-y(s)|+\delta]ds + \sum_{t_0\le\theta_i<t}[L|x(\theta_i)-y(\theta_i)|+\delta]$$

By using the corollary (3.1) of the G.B lemma for IDE given in (3.4), we get

$$|x(t)-y(t)| \le \frac{\delta}{L}[e^{LT}(1+L)^P-1] < \varepsilon$$

Remark 4.1.

If $g(t,y)=0, W_i(y)$=0, then in the above proof we get

$$|x(t)-y(t)| \le \int_{t_0}^{t}[L|x(s)-y(s)|]ds + \sum_{t_0\le\theta_i<t}[L|x(\theta_i)-y(\theta_i)|]$$

Which is the GB for IDE with $C=0$, therefore, we obtain

$$|x(t)-y(t)| \le 0 \text{ or } x(t)=y(t)$$

which also prove the unquiesces

4.2: Periodic Systems

Consider the IDE

$$\left.\begin{array}{r} x' = f(t,x), t \neq \theta_i \\ \Delta x|_{t=\theta_i} = I_i(x) \end{array}\right\} \qquad (4.3)$$

This system is said to be periodic if

P1) $\exists \omega \in R^+ \text{ such that } f(t+\omega,x)=f(t,x)\ \forall\ (t,x)\in R\times R^n$

P2) $\exists P \in Z^+ \text{ such that } I_{i+P}(x)=I_i(x)\ \forall\ x\in R^n$

P3) $\theta_{i+P}=\theta_i+\omega$ for all $i\in Z$

In this case, we say that the system is called $(\omega,P)-$ Periodic

Example 4.1.

The system

$x' = x^3\sin^2 t,\ \ t\neq\theta_i \,\&\, \Delta x|_{t=\theta_i}=(-1)^i x$, where $\theta_i=\frac{i\pi+1}{2}$, $i\in Z$ is $(\pi,2)$ periodic since

P1) $f(t+\pi,x)=x^3\sin^2(t+\pi)=x^3(-sint)^2=x^3\sin^2 t=f(t,x)\ \ \forall\ (t,x)\in R\times R^n$

P2) $I_{i+2}(x)=(-1)^{i+2}x=(-1)^{i+}x=I_i(x)\ P=2,4,6,\ldots$

P3) $\theta_{i+2}=\frac{(i+2)\pi+1}{2}=\frac{i\pi+1}{2}+\pi=\theta_i+\pi$ for all $i\in Z$

Lemma 4.1.

If $x(t)$ is a solution of equation (4.3), then $x(t+\omega)$ is also a solution

Proof

Suppose that: $x(t)$ is a solution of (4.3), let $\varphi(t) = x(t+\omega)$ we need to show that $\varphi\ (t)$is also solution of (4.3)

$$\varphi'(t) = x'(t+\omega) = f(t+\omega, x(t+\omega)) = f(t+\omega, \varphi(t))$$

By P1

$$\varphi'(t) = f(t, \varphi(t))$$

Then,$\varphi(t)$ implies the DE, moreover

$$\Delta\varphi|_{t=\theta_i} = \varphi(\theta_i^+) - \varphi(\theta_i^-) = x(\theta_i + \omega^+) - x(\theta_i + \omega) \quad \text{by P3}$$

$$= x(\theta_{i+P^+}) - x(\theta_{i+P}) = I_{i+P}(x(\theta_{i+P})) \quad \text{by P2}$$

$$= I_i(x(\theta_{i+P})) \quad \text{by P3}$$

$$= I_i\big(x(\theta_i + \omega)\big) = I_i\big(\varphi(\theta_i)\big)$$

Show that $\varphi(t)$ satisfies the impulse condition as well, therefore $\varphi(t)$ is also a solution of (4.3).

Example 4.2.

$$x' = 5x + sint, x(0) = x_0$$

$x_c = Ce^{5t}, x_p = ASint + BCost, A = -\frac{5}{26}, B = -\frac{1}{26}$

General solution

$$x(t) = Ce^{5t} - \frac{5}{26}Sint - \frac{1}{26}Cost$$

$$x(0) = C - \frac{1}{26} = x_0$$

therefore

$$x(t) = (x_0 + \frac{1}{26})e^{5t} - \frac{5}{26}Sint - \frac{1}{26}Cost$$

Theorem: Poincare Criteria (4.2) [15]**:**
Suppose that (4.3) has a unique solution for any initial condition. Then $x(t)$ is a ω −periodic solution of (4.3) iff $x(0) = x(\omega)$

Proof

⇒If $x(t)$ is a ω −periodic solution then

$x(t+\omega) = x(t)$ for all t in particular $t = 0$, we have

$$x(0) = x(\omega)$$

$\Leftarrow$ Suppose that $x(0) = x(\omega) = x_0$, let $x(t) = x(t, 0, x_0)$ be solution of (4.3). Then by the lemma we know that $\varphi(t) = x(t+\omega)$ is also a solution of (7) $\varphi(0) = x(\omega) = x(0) = x_0$ by the uniqueness, we have $\varphi(t) = x(t)$ for all t, that is $x(t+\omega) = x(t)$ for all t, this means that $x(t)$ is a periodic solution.

Example 4.3. If

$$x' = 2x, \ \ t \neq \theta_i \&$$

$$\Delta x|_{t=\theta_i} = 3 \ \text{ where } \theta_i = i + \frac{1}{2}$$

Is there 1-periodic solution?

Solution

Note that, here ω and P can be any positive integer. To have 1-Periodic solution by the Pervious Theorem. We need $x(0) = x(1)$. Let $x(0) = x_0$ on $[0, \frac{1}{2}]$, we have $x(t) = x_0 e^{2t}$and hence, $x\left(\frac{1}{2}\right) = x_0 e$, Now $x\left(\frac{1}{2}^{+}\right) = 3 + x_0 e$, on $(\frac{1}{2}, \frac{3}{2}]$. We have $x(t) = (3 + x_0 e)e^{2(t-\frac{1}{2})}$, Thus $x(1) = (3 + x_0 e)e$. Therefore $x(0) = x(1)$ iff $x_0 = (3 + x_0 e)e$.

Hence,

$$x_0 = \frac{3e}{1 - e^2},$$

that is the solution $x(t, t_0, x_0)$ is 1- Periodic solution

5. Linear homogeneous systems.

Homogenous system, if we consider the IDE

$$\left.\begin{aligned} x' &= A(t)x, t \neq \theta_i, \ x \in IR^n \\ \Delta x|_{t=\theta_i} &= B_i x(\theta_i) \end{aligned}\right\} \qquad (5.1)$$

Where $\{\theta_i\} \subset$ R and $\theta_i < \theta_{i+1}$ $\forall\, i \in Z$, $A(t)$ is continuous and bounded (n, n) Matrix function on (a, b), $B_i$is constant matrix for each $i \in Z$

Theorem 5.1 [1]**:**

$\forall\, x_0 \in IR^n$, there is a unique solution $x(t) = x(t, t_0, x_0)$ define for all $t \geq t_0$, moreover det $(I + B_i) \neq 0$ for all $i \in Z$, then this solution can be continued to $-\infty$.

Proof

Suppose that $\theta_i < t_0 < \theta_{i+1}$ for some $i \in Z$, then on $[t_0, \theta_{i+1}]$, $x(t) = x(t, t_0, x_0)$ exists as solution of the IVP

$$x' = A(t)x, \ x(t_0) = x_0$$

Then by the impulse condition, we have

$$x(\theta_{i+1}^{+}) - x(\theta_{i+1}) = B_i x(\theta_{i+1}) \Longrightarrow x(\theta_{i+1}^{+}) = (I + B_i)\, x(\theta_{i+1})$$

By induction $x(t)$ exists and unique on $[t_0, \infty)$. To proceed to $-\infty$ on interval $(\theta_i, t_0]$ $x(t)$ is the solution of

$$x' = A(t)x$$

$$x(t_0) = x_0$$

Now the problem is to find $x(\theta_i)$ since

$$x(\theta_i^{+}) = (I + B_i)x(\theta_i)$$

As

$$x(\theta_i) = (I + B_i)^{-1}x(\theta_i^{+})$$

By similar arguments, we have the unique solution defined on R.

Theorem 5.2 [1]:

The solution of linear homogeneous systems (5.1) from an n-dimensional linear vector space

Proof

Let $\varphi_1(t)$ and $\varphi_2(t)$ be two solutions of (5.1), then for any $C_1, C_2 \in R$, we have, when $t \neq \theta_i$

$$\begin{aligned}(C_1\varphi_1(t) + C_2\varphi_2(t))' = C_1\varphi'_1(t) + C_2\varphi'_2(t) = C_1A(t)\varphi_1(t) + C_2A(t)\varphi_2(t) \\ = A(t)(C_1\varphi_1(t) + C_2\varphi_2(t))\end{aligned}$$

Which means that $\boldsymbol{C_1\varphi_1(t)} + \boldsymbol{C_2\varphi_2(t)}$ is also solution of differential part at $\boldsymbol{t \neq \theta_i}$

Now for $\boldsymbol{t = \theta_i}$

$$\begin{aligned}\Delta\big(\boldsymbol{C_1\varphi_1(t) + C_2\varphi_2(t)}\big)|_{t=\theta_i} = \big[\big(\boldsymbol{C_1\varphi_1(\theta_i^{+}) + C_2\varphi_2(\theta_i^{+})}\big) - \big(\boldsymbol{C_1\varphi_1(\theta_i) + C_2\varphi_2(\theta_i)}\big)\big] \\ = \boldsymbol{C_1\big(\varphi_1(\theta_i^{+}) - \varphi_1(\theta_i)\big) + C_2\big(\varphi_2(\theta_i^{+}) - \varphi_2(\theta_i)\big) = C_1B_i\varphi_1(\theta_i) + C_2B_i\varphi_2(\theta_i)} \\ = \boldsymbol{B_i(C_1\varphi_1(\theta_i) + C_2\varphi_2(\theta_i))}\end{aligned}$$

Which means that $\boldsymbol{C_1\varphi_1(t) + C_2\varphi_2(t)}$ satisfies the impulse part, therefore, the set of the solution is a vector space, we need to show that this space is n-dimensional. Let $\{\boldsymbol{e_1, e_2, \dots, e_n}\}$ be standard basis for $\boldsymbol{R^n}$,

let $\boldsymbol{\varphi_j(t) = x(t, t_0, e_j)}$, $\boldsymbol{J = 1, 2, \dots, n}$ be the solution of $(\mathbf{5.1})$ satisfying $\boldsymbol{\varphi_j(t_0) = e_j}$, we will show that $\{\boldsymbol{\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)}\}$ is a basis for the solution space of $(\mathbf{5.1})$. Span. Let $\boldsymbol{x(t) = x(t, t_0, x_0)}$ be any solution of $(\mathbf{5.1})$ with $\boldsymbol{x(t_0) = x_0}$, clearly, there exist $\boldsymbol{C_1, C_2, \dots, C_n}$ such that

$$\mathbf{x_0 = C_1e_1 + C_2e_2 + \cdots C_ne_n}$$

Let

$$\boldsymbol{\varphi(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + \cdots + C_n\varphi_n(t)}$$

Clearly, $\boldsymbol{\varphi(t)}$ is a solution of $(\mathbf{5.1})$ and

$$\varphi(t_0) = C_1\varphi_1(t_0) + C_2\varphi_2(t_0) + \cdots + C_n\varphi_n(t_0) = C_1e_1 + C_2e_2 + \cdots C_ne_n = x_0 = x(t_0)$$

Means that

$\varphi(t) = x(t)\ \forall\, t \in IR$ because of the uniqueness that is $x(t)$ is a linear combination of $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$ since $x(t)$ is an arbitrary solution $\{\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)\}$ spans the solution set

Independence. Assume that

$$C_1\varphi_1(t) + C_2\varphi_2(t) + \cdots + C_n\varphi_n(t) = 0 \quad \forall\, t \in R$$

For some constants $C_1, C_2, \dots, C_n \in IR$. For $t = t_0$, we get

$$C_1\varphi_1(t_0) + C_2\varphi_2(t_0) + \cdots + C_n\varphi_n(t_0) = 0 \Longrightarrow C_1e_1 + C_2e_2 + \cdots C_ne_n = 0 \Longrightarrow \quad C_1 = C_2 = C_3 = \cdots = C_n = 0$$

Thus $\{\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)\}$ is linear independent. In this case, we can find another set of solution

$\{x_1(t), x_2(t), \dots, x_n(t)\}$ which is a basis. Hence the matrix

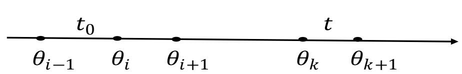
$$X(t) = \{x_1(t), x_2(t), \dots, x_n(t)\}$$

Is called a fundamental matrix, notice that $X(t)$ is a matrix solution of linear homogenous system and $det\big(X(t)\big) \neq 0\ \forall\, t \in R$

Let $t \neq \theta_i,\ x \in R^n$ and consider IDE

$$\left.\begin{aligned} x' &= Ax, t \neq \theta_i \\ \Delta x|_{t=\theta_i} &= Bx \end{aligned}\right\}$$

Where $AB = BA$ and let $det\,(I + B) \neq 0$. Let $x(t_0) = x_0$and say $t_0 \in (\theta_{i-1}, \theta_i)$ for some $i \in$Z



On $[0, \theta_i]$, we have $x(t)$ as solution of $x' = Ax,\ x(t_0) = x_0$

Which $x(t) = e^{A(t-t_0)}x_0$ and $x(\theta_i) = e^{A(\theta_i - t_0)}x_0$

$$x(\theta_i^+) = Bx(\theta_i) + x(\theta_i) = (I + B)x(\theta_i) = (I + B)e^{A(\theta_i - t_0)}x_0$$

On $(\theta_i, \theta_{i+1}]$,

$$x(t) = e^{A(t-\theta_i^+)}x(\theta_i^+) = e^{A(t-\theta_i^+)}(I + B)e^{A(\theta_i - t_0)}x_0.$$

Note that $AB = BA$ implies that

$$(I + B)e^{A(\theta_i - t_0)} = e^{A(\theta_i - t_0)}(I + B)$$

Therefore,

$$x(t) = e^{A(t-\theta_i)}e^{A(\theta_i - t_0)}(I + B)x_0 = e^{A(t-t_0)}(I + B)x_0$$

On $(\theta_{i+1}, \theta_{i+2}]$,

$$x(t) = e^{A(t-\theta_{i+1}^{+})}(I+B)x(\theta_{i+1}^{+}) = e^{A(t-\theta_{i+1}^{+})}(I+B)e^{A(\theta_{i+1}-t_0)}(I+B)x_0$$

Therefore,

$$x(t) = e^{A(t-t_0)}(I+B)^2 x_0$$

For $(\theta_k, \theta_{k+1}]$,

$$x(t) = e^{A(t-t_0)}(I+B)^{k-i+1} x_0$$

In general

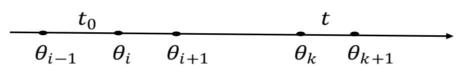
$$x(t) = e^{A(t-t_0)}(I+B)^{i(t_0,t)} x_0$$

Where $i(t_0, t)$ denote the number of θ_i on interval (t_0, t)

Example 5.2.

Let $t \neq \theta_i$, $x \in IR^n$ and consider IDE

$$\left.\begin{aligned} x' &= Ax, t \neq \theta_i \\ \Delta x|_{t=\theta_i} &= Bx \end{aligned}\right\} \qquad (5.2)$$



$$x(t, t_0, x_0) e^{A(t-\theta_k)}(I+B)e^{A(\theta_k-\theta_{k-1})} \ldots e^{A(\theta_{i+1}-\theta_i)}(I+B)e^{A(\theta_i-t_0)}x_0$$

If $AB = BA$ then

$$x(t, t_0, x_0) = e^{A(t-t_0)}(I+B)^{i(t_0,t)} x_0$$

Where $i(t_0, t)$ denote the number of θ_i on interval (t_0, t)

$$x(t) = e^{\lambda(t-t_0)}(I+B)^{i(t_0,t)} v_1$$

Is the solution of impulse system

$$\left.\begin{aligned} x' &= Ax, t \neq \theta_i \\ \Delta x|_{t=\theta_i} &= Bx \end{aligned}\right\}$$

Verifying this claim

If we take any solution

$$x(t, t_0, x_0) = e^{A(t-t_0)} x_0$$

$$x' = Ax$$

$$x_J(t) = e^{\lambda_J(t-t_0)} v_J$$

If $Re\lambda_J(A) < 0$ for all J, the solution is asymptotically stable

If $Re\lambda_J(A) > 0$ for some J, the solution is unstable

If $Re\lambda_J(A) = 0$ for all J, the solution is stable

Whereas for IDE, we note that

$$x(t, t_0, x_0) = e^{A(t-t_0)}(I+B)^{i(t_0,t)} x_0$$

Can be written as follows

$$x(t, t_0, x_0) = e^{A(t-t_0)+i(t_0,t)\ln(I+B)} x_0$$

$$= e^{[A+P\,ln(I+B)](t-t_0)}x_0.$$

Where $P = \frac{i(t_0,t)}{(t-t_0)}$

Observation, let $\Lambda = A + P\,ln(I + B)$ and $\lambda_J = \lambda_J(\Lambda)$

i) If $Re\lambda_J < 0$ for all J, then the trivial solution of (5.2) is asymptotically stable

ii) If $Re\lambda_J \leq 0$ for all J, and λ_J is simple, the solution is stable

iii) If **there** is an eigenvalue with positive real part then solution is unstable

Consider the IDE

$$\left.\begin{array}{l} x' = Ax, t \neq \theta_i,\ x \in R^n \\ \Delta x|_{t=\theta_i} = B_i x(\theta_i) \end{array}\right\} \qquad (5.3)$$

Suppose that $u(t,s)$ is the transition matrix of $x' = A(t)x$ [that $u(t,s)$ is a matrix solution and $u(s,s) = I$]. [in this case $x(t,t_0,x_0) = u(t,t_0,x_0)$]

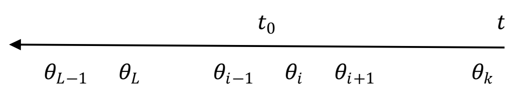
Then $x(t,t_0,x_0)$?

$$x(t,t_0,x_0) = u(t,\theta_k)(I + B_k)u(\theta_k,\theta_{k-1})(I + B_{k-1}) \ldots u(\theta_{i+2},\theta_{i+1})(I + B_{i+1})u(\theta_{i+1},\theta_i)(I + B_i)u(\theta_i,t_0)x_0$$

For $t \in (\theta_k,\theta_{k+1})$, on $[t_0,\theta_i]$ $x(t) = u(t,t_0)x_0$, Just think about x_0 to prove that

$$x(t,t_0) = u(t,\theta_k)(I + B_k) \ldots u(\theta_{i+1},\theta_i)(I + B_i)u(\theta_i,t_0)x_0$$

Is called the transition matrix of (5.3) for this problem.



We can prove it by induction when $t \in [t_0,\theta_i]$, then, $x(t,t_0,x_0) = e^{A(t-t_0)}x_0 = U(t,t_0)x_0$

We know that

$$x(\theta_i,t_0,x_0) = e^{A(\theta_i-t_0)}x_0 = U(\theta_i,t_0)x_0,$$

we want to find $x(\theta_i^+,t_0,x_0)$.

$$x(\theta_i^+,t_0,x_0) = (I + B_i)x(\theta_i,t_0,x_0) = (I + B_i)e^{A(\theta_i-t_0)}x_0 = (I + B_i)U(\theta_i,t_0)x_0$$

When $t \in (\theta_i,\theta_{i+1}]$, then, we have that

$$x' = Ax$$

And

$$x(\theta_i^+,t_0,x_0) = (I + B_i)e^{A(\theta_i-t_0)}x_0.$$

We get that,

$$x(t,t_0,x_0) = e^{A(t-\theta_i^+)}X(\theta_i^+) = e^{A(t-\theta_i^+)}(I + B_i)e^{A(\theta_i-t_0)}x_0 = U(t,\theta_i)(I + B_i)U(\theta_i,t_0)x_0.$$

We know that

$$x(\theta_{i+1}, t_0, x_0) = e^{A(\theta_{i+1}-\theta_i^+)}(I + B_i)e^{A(\theta_i - t_0)}x_0 = U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0,$$

we want to find $x(\theta_{i+1}^+, t_0, x_0)$.

$$x(\theta_{i+1}^+, t_0, x_0) = (I + B_{i+1})x(\theta_{i+1}, t_0, x_0) = (I + B_{i+1})e^{A(\theta_{i+1}^+-\theta_i^+)}(I + B_i)e^{A(\theta_i - t_0)}x_0$$
$$= (I + B_{i+1})U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0$$

Now, let us assume that the system is true when $t \in (\theta_{k-1}, \theta_k]$, and we want to prove it when $t \in (\theta_k, \theta_{k+1}]$.

That means, we have from system

$$x(t, t_0, x_0) = e^{A(t-\theta_{k-1}^+)}x(\theta_{k-1}^+) = U(t, \theta_{k-1})(I + B_{k-1})U(\theta_{k-1}, \theta_{k-2}) \dots u(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0$$

We know that

$$x(\theta_k, t_0, x_0) = e^{A(\theta_k-\theta_{k-1}^+)}\, x(\theta_k, t_0, x_0)$$
$$= e^{A(\theta_k-\theta_{k-1}^+)}(I + B_{k-1})U(\theta_{k-1}, \theta_{k-2}) \dots U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0$$
$$= U(\theta_k, \theta_{k-1})(I + B_{k-1})U(\theta_{k-1}, \theta_{k-2}) \dots U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0$$

And we also can get

$$x(\theta_k^+, t_0, x_0) = (I + B_k)x(\theta_k, t_0, x_0)$$
$$= (I + B_k)U(\theta_k, \theta_{k-1})(I + B_{k-1})U(\theta_{k-1}, \theta_{k-2}) \dots U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0.$$

Therefore, we have

$$x' = Ax$$

and

$$x(\theta_k^+, t_0, x_0) = (I + B_k)U(\theta_k, \theta_{k-1})(I + B_{k-1})U(\theta_{k-1}, \theta_{k-2}) \dots U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0$$

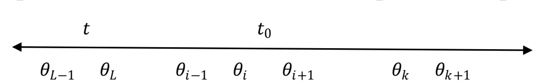
when $t \in (\theta_k, \theta_{k+1}]$. Then,

$$x(t, t_0, x_0) = e^{A(t-\theta_k)}x(\theta_k^+)$$
$$= U(t, \theta_k)(I + B_k)U(\theta_k, \theta_{k-1})(I + B_{k-1})u(\theta_{k-1}, \theta_{k-2}) \dots U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0$$

For all $t \geq t_0$

We want to find d $x(t, t_0, x_0)$ for $t < t_0$.

It means that we have to proceed back the same steps in the previous case.



We can prove it by induction, when $t \in [\theta_i, t_0]$, then

$\boldsymbol{x(t,t_0,x_0) = e^{A(t-t_0)}x_0 = U(t,t_0)x_0}$

We have that,

$\boldsymbol{x(\theta_i^+,t_0,x_0) = e^{A(\theta_i^+-t_0)}x_0 = U(\theta_i,t_0)x_0}$.

Now, we need to find $\boldsymbol{X(\theta_i,t_0,x_0)}$ by using impulse condition, we obtain

$$\boldsymbol{x(\theta_i,t_0,x_0) = (I+B_i)^{-1}x(\theta_i^+,t_0,x_0) = (I+B_i)^{-1}e^{A(\theta_i^+-t_0)}x_0 = (I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

We have that,

$$\boldsymbol{x' = Ax}$$

$$\boldsymbol{x(\theta_i,t_0,x_0) = (I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

Now when $\boldsymbol{t \in (\theta_{i-1},\theta_i]}$, we have

$$\boldsymbol{x(t,t_0,x_0) = e^{A(t-\theta_i)}x(\theta_i,t_0,x_0) = e^{A(t-\theta_i)}(I+B_i)^{-1}U(\theta_i,t_0)x_0 = U(t,\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

We have that, $\boldsymbol{x(\theta_{i-1}^+,t_0,x_0)}$ and by using impulse condition, we can get $\boldsymbol{x(\theta_{i-1},t_0,x_0)}$

$$\boldsymbol{x(\theta_{i-1},t_0,x_0) = (I+B_{i-1})^{-1}x(\theta_{i-1}^+,t_0,x_0) = (I+B_{i-1})^{-1}e^{A(\theta_{i-1}^+-\theta_i)}(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$
$$\boldsymbol{= (I+B_{i-1})^{-1}U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

We have that,

$$\boldsymbol{x' = Ax}$$

$$\boldsymbol{x(\theta_{i-1},t_0,x_0) = (I+B_{i-1})^{-1}U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

Now when $\boldsymbol{t \in (\theta_{i-2},\theta_{i-1}]}$, we have

$$\boldsymbol{x(t,t_0,x_0) = e^{A(t-\theta_{i-1})}x(\theta_{i-1},t_0,x_0) = e^{A(t-\theta_{i-1})}(I+B_{i-1})^{-1}U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

$$\boldsymbol{= U(t,\theta_{i-1})(I+B_{i-1})^{-1}U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

We have that, $\boldsymbol{x(\theta_{i-2}^+,t_0,x_0)}$ and by using impulse condition, we can get $\boldsymbol{x(\theta_{i-2},t_0,x_0)}$

$$\boldsymbol{x(\theta_{i-2},t_0,x_0) = (I+B_{i-2})^{-1}x(\theta_{i-2}^+,t_0,x_0)}$$
$$\boldsymbol{= (I+B_{i-2})^{-1}e^{A(\theta_{i-2}^+-\theta_{i-1})}U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$
$$\boldsymbol{= (I+B_{i-2})^{-1}U(\theta_{i-2},\theta_{i-1})(I+B_{i-1})^{-1}U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

Now, let us assume that the system (5.3) implies that, when $\boldsymbol{t \in (\theta_L,\theta_{L+1}]}$,

$$\boldsymbol{x(t,t_0,x_0) = e^{A(t-\theta_{L+1})}x(\theta_{L+1},t_0,x_0) = e^{A(t-\theta_{L+1})}(I+B_{L+1})^{-1}U(\theta_{L+1},\theta_{L+2})\dots U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

$$\boldsymbol{= U(t,\theta_{L+1})(I+B_{L+1})^{-1}U(\theta_{L+1},\theta_{L+2})\dots U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

and also, we have that. $\boldsymbol{x(\theta_L^+,t_0,x_0)}$ and by using impulse condition, we can get $\boldsymbol{x(\theta_L,t_0,x_0)}$, by using impulse condition, we obtain.

$$\boldsymbol{x(\theta_L,t_0,x_0) = (I+B_L)^{-1}x(\theta_L^+,t_0,x_0) = (I+B_L)^{-1}U(\theta_L,\theta_{L+1})(I+B_{L+1})^{-1}U(\theta_{L+1},\theta_{L+2})\dots U(\theta_{i-1},\theta_i)(I+B_i)^{-1}U(\theta_i,t_0)x_0}$$

We want to find the solution of system 5.3, when $t \in (\theta_{L-1}, \theta_L]$, we have

$$x' = Ax$$

$$x(\theta_L, t_0, x_0) = (I + B_L)^{-1}U(t, \theta_{L+1})\,(I + B_{L+1})^{-1}U(\theta_{L+1}, \theta_{L+2}) \dots U(\theta_{i-1}, \theta_i)(I + B_i)^{-1}U(\theta_i, t_0)x_0$$

Therefore,

$$x(t, t_0, x_0) = e^{A(t-\theta_L)}x(\theta_L, t_0, x_0)$$

$$= e^{A(t-\theta_L)}\,(I + B_L)^{-1}U(\theta_L, \theta_{L+1})\,(I + B_{L+1})^{-1}U(\theta_{L+1}, \theta_{L+2}) \dots U(\theta_{i-1}, \theta_i)(I + B_i)^{-1}U(\theta_i, t_0)x_0$$

$$= U(t, \theta_L)(I + B_L)^{-1}U(\theta_L, \theta_{L+1})\,(I + B_{L+1})^{-1}U(\theta_{L+1}, \theta_{L+2}) \dots U(\theta_{i-1}, \theta_i)(I + B_i)^{-1}U(\theta_i, t_0)x_0$$

for all $t < t_0$.

We can write the solution of the system (5.3) as piecewise function

$$x(t, t_0, x_0) = \begin{cases} U(t, \theta_k)(I + B_k)U(\theta_k, \theta_{k-1}) \dots U(\theta_{i+1}, \theta_i)(I + B_i)U(\theta_i, t_0)x_0 & t \geq t_0 \\ I & \\ U(t, \theta_L)(I + B_L)^{-1}U(\theta_L, \theta_{L+1}) \dots U(\theta_{i-1}, \theta_i)(I + B_i)^{-1}U(\theta_i, t_0)x_0 & t < t_0 \end{cases}$$

Important notice, if $AB = BA$, hence the solution will be as

$$x(t, t_0, x_0) = \left\{ \begin{matrix} U(t, t_0)\prod_{m=i}^{k}(I + B_m)\,x_0 & if\ \ t \geq t_0 \\ I & if\ \ t = t_0 \\ U(t, t_0)\prod_{n=i}^{L}(I + B_n)^{-1}\,x_0 & if\ t < t_0 \end{matrix} \right\}$$

In this case $x(t, t_0, x_0) = x(t, t_0\,)x_0$ may also be called as the matriciant.

Suppose $X(t, t_0\,) = [q_{ij}(t)]\, i, j = 1, 2, \dots, n,\ t \geq t_0$

For any solution $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ we have

$$X(t, t_0, y_0) - X(t, t_0, x_0) = X(t, t_0\,)(y_0 - x_0)$$

$$\left. \begin{matrix} x' = Ax, t \neq \theta_i \\ \Delta x|_{t=\theta_i} = B_i x(\theta_i) \end{matrix} \right\} \qquad (5.4)$$

i) If $X(t, t_0\,)$ or any fundamental matrix is bounded for $t \geq t_0$, then

$$\|X(t, t_0\,)\| = \sum_{i,j=1}^{n} |q_{ij}(t)| \leq M < \infty$$

For some M, we have

$$\|X(t, t_0, y_0) - X(t, t_0, x_0)\| \leq \|X(t, t_0\,)\|\|y_0 - x_0\| \leq M\|y_0 - x_0\|$$

Now for any $\varepsilon > 0$, if we choose $\delta < \frac{\varepsilon}{M}$, then

$\|y_0 - x_0\| < \delta$ implies $\|X(t, t_0, y_0) - X(t, t_0, x_0)\| \leq \varepsilon$

That is, the solution $X(t, t_0, x_0)$ of (5.3) is stable

ii) If $\lim_{t\to\infty} \|X(t, t_0)\| = 0$, then clearly $X(t, t_0)$ is bounded for $t \geq t_0$ and hence (5.3) is stable, moreover

$\lim_{t\to\infty} \|X(t, t_0, y_0) - X(t, t_0, x_0)\| = 0 \text{ for all } x_0, y_0$. Therefore $X(t, t_0, x_0)$ or any solution is asymptotically stable

$$X(t, t_0)x_0 = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} x_{10} \\ \vdots \\ x_{n_0} \end{bmatrix} = \begin{bmatrix} \ddots \\ \ddots \\ \ddots \end{bmatrix}$$

iii) If $X(t, t_0)$ is unbounded then $q_{ij}(t)$ is unbounded for some i, j then i^{th}component of $X(t, t_0, x_0)$ is

$$X(t, t_0, x_0) = q_{i1}(t)x_{10} + \cdots + q_{in}x_{n0}$$

Let y_0 be such that

$y_{1,0} = x_{10}, \ldots y_{j,0} \neq x_{j,0}, y_{j+1,0} = x_{j+1,0} = \cdots y_{n0} = x_{n0}$

Then

$$\lim_{t\to\infty} \|X(t, t_0, y_0) - X(t, t_0, x_0)\| = \lim_{t\to\infty} \|q_{ij}(t)\| \|y_{j,0} - x_{j,0}\| = \infty$$

For all $y_{j,0} \neq x_{j,0}$ therefore (5.3) is unstable

Theorem 5.3 [15]:

The system (5.4) is said to be stable, asymptotically stable and unstable iff $X(t, t_0)$ is bounded $\lim_{t\to\infty} \|X(t, t_0)\| = 0$ and $X(t, t_0)$ is unbounded respectively

Example 5.3.

For which $k \in IR$, the system

$x'_1 = -2x_2$ & $x'_2 = 2x_1, t \neq \theta_i$

$\Delta x_1|_{t=\theta_i} = kx_1$ & $\Delta x_2|_{t=\theta_i} = kx_2$

Is Stable, unstable and asymptotically stable?

It is a clear that

$$\phi(t, s) = \begin{pmatrix} cos2(t-s) & -sin2(t-s) \\ sin2(t-s) & cos\,(2(t-s)) \end{pmatrix}$$

This is a fundamental matrix solution and $\phi(s, s) = I$ is transition matrix

$x(t, t_0) = \phi(t, \theta_p)(I + B_p)\, \phi(\theta_p, \theta_{p-1})(I + B_{p-1}) \ldots \phi(\theta_{i+1}, \theta_i)(I + B_i)$

$$\phi(\theta_i, t_0) = \begin{pmatrix} cos2(t-\theta_p) & -sin2(t-\theta_p) \\ sin2(t-\theta_p) & cos\,(2(t-\theta_p) \end{pmatrix} \begin{pmatrix} 1+k & 0 \\ 0 & 1+k \end{pmatrix} \dots \begin{pmatrix} cos2(\theta_i-t_0) & -sin2(\theta_i-t_0) \\ sin2(\theta_i-t_0) & cos\,(2(\theta_i-t_0) \end{pmatrix}$$

$$= (1+k)^{i(t_0,t)} \begin{pmatrix} cos2(t-t_0) & -sin2(t-t_0) \\ sin2(t-t_0) & cos\,(2(t-t_0) \end{pmatrix}$$

i) if $|1+k| < 1 \Rightarrow -2 < k < 0$, then, $\lim_{t\to\infty} \|x(t,t_0)\| = 0$, thus the system is asymptotically stable

ii) if $|1+k| > 1 \Rightarrow k > 0$ or $k < -2$, then the system is an unstable

iii) if $|1+k| = 1$, ($k = 0$ *or* $k = -2$), then the system is a stable.

6. Conclusions

This paper focuses on the Impulsive differential equations and describe the Gron-wall-Bellman Lemma and conditions of periodic systems, and also, we obtained some important results about Linear and stability of IDE. The future suggestion to study Impulsive dynamic equations on time scales, in recent years dynamic equations on time scales have received much attention. The time scales calculus has a massive potential of biotechnology and mathematical models, which we will focus on it in future studies.

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