

Construction and Classification of Exact Jacobi Elliptic Traveling-Wave Solutions for the Coupled Whitham–Broer–Kaup System via the Generalized F-Expansion Method

Ragab M. A. Almasroub¹ and Ramadan M. Al-Naas Al-Wahishi²

Department of Mathematics, Faculty of Science, Azzaytuna University, Tarhuna, Libya

بناء وتصنيف حلول الموجة الاهليجية الجاكوبية الدقيقة لنظام ويثام-بروير-كاب المقترن عبر طريقة التوسيع F المعممة

almasroub4392@gmail.com

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Abstract

This study investigates the nonlinear coupled Whitham–Broer–Kaup (WBK) system, a prototypical model for dispersive shallow-water wave propagation, where $u(x, t)$ denotes the horizontal fluid velocity and $v(x, t)$ the free-surface elevation. The generalized F-expansion method, embedded in the Jacobi elliptic-function framework, is used to derive exact traveling-wave solutions. Setting $\xi = x - ct$ reduces the system to a single second-order ordinary differential equation (ODE). The integration constants are fixed at $C_1 = C_2 = 0$; this is the zero-background integration-constant subcase, which simplifies the reduced system, not a pointwise boundary condition imposed on the periodic profiles themselves, which remain regular classical solutions on pole-truncated domains and degenerate to smooth localized forms only along special branches (Remark 1). A balanced rational ansatz in an auxiliary function $F(\xi)$ obeying a quartic first-order ODE yields ten algebraically independent exact closed-form solution representations: five are real-valued throughout their stated parameter range, four are complex-valued formal branches for every modulus, and one (Case 4) is complex for most of the modulus range but real on a previously unreported complementary sub-range once a correlated amplitude-sign choice is made (Remark 5); five further representations, obtained from reciprocal Jacobi substitutions applied to these, are algebraically identical to representations already counted and are recorded, but not separately tabulated (Table 2). Every family is verified by direct symbolic substitution into the reduced ODE (Appendix A). We show that the composite (two-sided) rational families, Cases 7, 8,9,10, are structurally independent of the elliptic equation rational expansion method of [Chen et al. \(2005\)](#) – whose auxiliary equation is likewise quartic and whose enumeration already contains the same composite auxiliary functions – via a boundedness argument on the two methods' velocity-field ansatz (Lemma 1). Representative traveling-wave profiles, computed directly from the closed-

Corresponding author: almasroub4392@gmail.com¹

Email: naasramadan@gmail.com²

form expressions with explicit pole truncation, are shown in Section 7; in particular we correct an error present in an earlier treatment of this material and show that the solitary limit $m \rightarrow 1$ of Case 1 does *not* approach the classical smooth kink of [Xie et al. \(2001\)](#) – it remains singular at the zeros of sn – and that the smooth kink is instead a distinct degenerate branch of the underlying algebraic system. Finally, since the auxiliary equation (3) is biquadratic in F , every family in Table 1 is shown (Section 6.3) to admit, alongside its Jacobi form, an equivalent closed-form Weierstrass-elliptic representation, with explicit invariants g_2, g_3 given directly in terms of l_0, l_1, l_2 and verified both symbolically and numerically (Appendix C) – a unifying result that a further attempted extension of this material tried to reach but could not establish rigorously, since its extra branches relied on undeclared parameters not tied to the governing algebraic system (12). We further resolve the case $C_1 \neq 0$ left open above: every family in Table 1 extends to a one-parameter family with a nonzero background free-surface level, obtained by replacing c by $\tilde{c} = \sqrt{c^2 + 2C_1}$ in g_1, f_1 under the compatibility constraint $C_2 = -cC_1$ (Theorem 1), verified to 50 digits against the original, unintegrated PDE system. The present work establishes exact traveling-wave solutions only; no claim of dynamical or spectral stability is made, and stability is left for a separate analysis (Section 9).

Keywords: *Coupled Whitham–Broer–Kaup system, generalized F-expansion method, Jacobi elliptic functions, exact analytical solutions, nonlinear PDEs, traveling-wave reduction, periodic and solitary waves, shallow-water waves.*

المخلص

تتناول هذه الدراسة نظام ويزام-بروير-كاوب (WBK) المقترن غير الخطي، وهو نموذج نمطي لانتشار الموجات المشيئة في المياه الضحلة، حيث تمثل $u(x, t)$ السرعة الأفقية للمائع و $v(x, t)$ ارتفاع السطح الحر. وتستخدم طريقة التوسيع المعممة (F-expansion)، ضمن إطار دوال جاكوبي الإهليلجية، لاشتقاق حلول موجية متنقلة دقيقة. ويؤدي التعويض $\xi = x - ct$ إلى اختزال النظام إلى معادلة تفاضلية عادية واحدة من الرتبة الثانية. وتثبت ثوابت التكامل عند $C_1 = C_2 = 0$ ، وهي الحالة الجزئية ذات المستوى الخلفي الصفري لثوابت التكامل التي تبسط النظام المختزل، وليست شرطاً حدياً نقطياً مفروضاً على المقاطع الدورية نفسها؛ إذ تبقى هذه المقاطع حلولاً كلاسيكية منتظمة على مجالات مقطوعة الأقطاب، ولا تؤول إلى أشكال موضعية ملساء إلا على فروع خاصة (الملاحظة 1). ويُفرض افتراض كسري متوازن في دالة مساعدة $F(\xi)$ تحقق معادلة تفاضلية رباعية من الرتبة الأولى إلى عشرة تمثيلات مغلقة دقيقة للحلول، مستقلة جبرياً: خمسة منها حقيقية القيم على كامل مدى معاملاتها المعلن، وأربعة فروع شكلية مركبة القيم لكل قيمة للمعامل، وواحد (الحالة 4) مركب على معظم مدى المعامل لكنه حقيقي على مدى جزئي مكمل لم يذكر سابقاً عند اختيار إشارة مترابطة للسعة (الملاحظة 5)؛ أما التمثيلات الخمسة الإضافية الناتجة عن تطبيق التعويضات العكسية لدوال جاكوبي عليها فهي مطابقة جبرياً لتمثيلات محسوبة سلفاً، ولذا دُوت دون أن تُجدول مستقلة (الجدول 2). وقد تُحقق من كل عائلة بالتعويض الرمزي المباشر في المعادلة المختزلة (الملحق A). ونبين أن العائلات الكسرية المركبة (ثنائية الطرف)، أي الحالات 7 و8 و9 و10، مستقلة بنيوياً عن طريقة التوسيع الكسري للمعادلة الإهليلجية لـ [Chen et al. \(2005\)](#) – التي معادلتها المساعدة رباعية كذلك، ويتضمن تعادها الدوال المساعدة المركبة نفسها – وذلك عبر حجة المحدودية المطبقة على افتراض حقل السرعة في الطريقتين (التمهيدية 1). وتُعرض في القسم 7 مقاطع موجية تمثيلية محسوبة مباشرة من الصيغ المغلقة مع اقتطاع صريح للأقطاب؛ وعلى وجه الخصوص نصح خطاً ورد في معالجة سابقة لهذه المادة، ونبين أن النهاية المنعزلة $m \rightarrow 1$ للحالة 1 لا تقترب من موجة الانتقال الملساء الكلاسيكية

لـ (Xie et al., 2001) – إذ تبقى شاذة عند أصفار الدالة sn – وأن موجة الانتقال الملساء هي بدلاً من ذلك فرع منحل مستقل للنظام الجبري الأساسي. وأخيراً، بما أن المعادلة المساعدة (3) ثنائية التربيع في F ، نبين (القسم 6.3) أن كل عائلة في الجدول 1 تقبل، إلى جانب صيغتها الجاكوبية، تمثيلاً مغلقاً مكافئاً بدالة فايرشتراس الإهليلجية، مع ثابتي لا تتغير g_2, g_3 معطين صراحةً بدلالة l_0, l_1, l_2 ، ومُتحققَ منهما رمزياً وعددياً (الملحق C) – وهي نتيجة توحيدية حاولت محاولة لاحقة لتوسيع هذه المادة الوصول إليها لكنها لم تتمكن من إثباتها بدقة، لأن فروعها الإضافية اعتمدت على معاملات غير معروفة لا ترتبط بالنظام الجبري الحاكم (12). كما نحل الحالة $C_1 \neq 0$ التي تُركت مفتوحة أعلاه: إذ تمتد كل عائلة في الجدول 1 إلى عائلة ذات معامل واحد بمستوى خلفي غير صفري للسطح الحر، تُحصل باستبدال c بـ $\tilde{c} = \sqrt{c^2 + 2C_1}$ في g_1, f_1 تحت قيد التوافق $C_2 = -cC_1$ (النظرية 1)، مع التحقق منها بدقة 50 رقماً مقابل نظام المعادلات التفاضلية الجزئية الأصلي غير المُكامل. ويقتصر هذا العمل على إنشاء حلول موجية متنقلة دقيقة؛ فلا يُدعى فيه أي استقرار ديناميكي أو طيفي، ويُترك الاستقرار لتحليل مستقل (القسم 9).

الكلمات المفتاحية: نظام ويزام-بروير-كاوب المقترن؛ طريقة التوسيع المعممة (F -expansion)؛ دوال جاكوبي الإهليلجية؛ حلول تحليلية دقيقة؛ معادلات تفاضلية جزئية غير خطية؛ اختزال الموجة المتنقلة؛ الموجات الدورية والمنعزلة؛ موجات المياه الضحلة.

1 Introduction

Nonlinear partial differential equations (PDEs) model a wide spectrum of physical phenomena, including fluid dynamics, plasma physics, and nonlinear wave propagation in dispersive media (Abdoon, 2015; Almasroub et al., 2020; Maruno et al., 2003). Among the paradigmatic models, the Whitham–Broer–Kaup (WBK) equations provide a canonical framework for shallow-water wave dynamics, combining nonlinear convective effects with dispersive corrections: $u(x, t)$ is the horizontal velocity and $v(x, t)$ the free-surface displacement from the undisturbed water level (Xie et al., 2001). The system has been extensively studied for the formation, interaction, and stability of its nonlinear waves (Ali and Younas, 2025; El-Sabbagh and El-Ganaini, 2012; Kumar et al., 2023; Tariq et al., 2024; Xie et al., 2001; Chen et al., 2005; Xu et al., 2017; Zhang and Bilal, 2024).

A diverse array of analytical techniques has been developed for such equations: the first-integral method (Abdoon, 2015; El-Sabbagh and El-Ganaini, 2012), the Exp-function method (El-Borai et al., 2015; He, 2013), the functional variable method (Zerarka et al., 2010; Zerarka et al., 2016), the Jacobi elliptic function method (Gepreel, 2014; Maruno et al., 2003), the tanh-function method (El-Borai and Al-Masroub, 2015; Parkes, 2010; Wazwaz, 2007), the (G'/G) -expansion method (Almasroub et al., 2020; Kumar and Chand, 2014; Zheng, 2013), the $(G'/G, 1/G)$ -expansion method (Al-shawba et al., 2018), the Kudryashov method (Zayed and Alurrfi, 2014; Zayed and Alurrfi, 2015), the generalized Riccati mapping method (El-Borai et al., 2016; Li et al., 2014), and dynamical reduction approaches (Seadawy et al., 2025; Zhang et al., 2008).

Specifically for the WBK system, Xie et al. (2001) derived explicit traveling-wave solutions via an automated hyperbolic (tanh/sech) ansatz; El-Sabbagh and El-Ganaini (2012) applied the first-integral method, based on the Division Theorem for polynomial

rings, to extract periodic and solitary profiles; [Chen et al. \(2005\)](#) developed the *elliptic equation rational expansion method*, whose auxiliary function satisfies the general quartic relation $(\phi')^2 = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4$, and whose rational ansatz for the velocity field, once specialized to the balance required by the WBK system, takes the form $U(\eta) = a_0 + a_1\phi + \frac{b_1\phi'}{l\phi+1}$ – a form built from nonnegative powers of ϕ (and ϕ'), with no term of the form $1/\phi$; [Wang et al. \(2020\)](#) used generalized Riccati-equation techniques; [Zhang et al. \(2008\)](#) and [Xu et al. \(2017\)](#) applied symmetry analysis, Lie-group reduction, and Wronskian techniques.

Problem statement. The constructions above either produce single hyperbolic solitary waveforms ([El-Sabbagh and El-Ganaini, 2012](#); [Wang et al., 2020](#); [Xie et al., 2001](#)), algebraic branches from a quartic auxiliary equation whose associated rational ansatz is restricted to nonnegative powers of the auxiliary variable (and hence remains bounded, Mobius-type, as that variable diverges) ([Chen et al., 2005](#)), or interaction solutions of a different nature ([Xu et al., 2017](#); [Zhang et al., 2008](#)). What has been missing is a systematic taxonomy of composite rational Jacobi profiles built from a rational ansatz that explicitly includes the reciprocal term $1/F$ in addition to F itself, together with a rigorous proof that the resulting composite families are structurally new precisely because of this reciprocal term, and not a reparametrization of the solutions of [Xie et al. \(2001\)](#) or [Chen et al. \(2005\)](#).

Contributions.

- (i) Ten algebraically independent closed-form solution representations, five of which are real-valued throughout their declared parameter ranges and one (Case 4) real on a previously unreported sub-range under a correlated amplitude-sign correction, together with five further representations shown to be algebraic duplicates of these and omitted from the main classification for that reason;
- (ii) a rigorous, verifiable proof (Lemma 1) that the composite families 7,8,9,10 cannot be generated by the elliptic equation rational expansion method of [Chen et al. \(2005\)](#), notwithstanding the fact that their auxiliary equation is, like the present one, quartic, and that their enumeration already lists the same composite auxiliary functions;
- (iii) complete parameter constraints ensuring real-valued, pole-truncated, regular profiles;
- (iv) symbolic verification of every family (Appendix A) with a fully worked numerical cross-check (Appendix B);
- (v) representative traveling-wave profiles regenerated directly from the closed-form expressions, together with a corrected account of the $m \rightarrow 1$ limiting behavior of Case 1 (Section 7);
- (vi) a universal Weierstrass-elliptic representation (Section 6.3) valid for every family in Table 1 simultaneously, with closed-form invariants g_2, g_3 derived from first principles and verified symbolically and numerically (Appendix C);

(vii) the general $C_1 \neq 0$ family (Section 8), resolving the direction left open in Remark 1: a closed-form one-parameter extension of every family in Table 1 to a nonzero background free-surface level, verified to 50 digits against the original PDE system (Appendix D).

2 The Generalized F-Expansion Method

2.1 Jacobi elliptic functions

Let $m \in (0,1)$ be the elliptic modulus and $\text{sn}(\xi, m)$, $\text{cn}(\xi, m)$, $\text{dn}(\xi, m)$ the basic Jacobi elliptic functions, satisfying

$$\begin{aligned} \text{sn}' &= \text{cn} \text{dn}, & \text{cn}' &= -\text{sn} \text{dn}, & \text{dn}' &= -m^2 \text{sn} \text{cn}, & \text{cn}^2 + \text{sn}^2 &= 1, & \text{dn}^2 \\ &= 1 - m^2 \text{sn}^2. \end{aligned} \quad (1)$$

Following Glaisher, the remaining nine functions are

$$\begin{aligned} \text{ns} &= \frac{1}{\text{sn}}, & \text{nc} &= \frac{1}{\text{cn}}, & \text{nd} &= \frac{1}{\text{dn}}, & \text{sc} &= \frac{\text{sn}}{\text{cn}}, & \text{cs} &= \frac{\text{cn}}{\text{sn}}, & \text{sd} &= \frac{\text{sn}}{\text{dn}}, \\ \text{ds} &= \frac{\text{dn}}{\text{sn}}, & \text{cd} &= \frac{\text{cn}}{\text{dn}}, & \text{dc} &= \frac{\text{dn}}{\text{cn}}. \end{aligned} \quad (2)$$

As $m \rightarrow 1$ ($\text{sn}, \text{cn}, \text{dn}$) $\rightarrow$ ($\tanh, \text{sech}, \text{sech}$), as $m \rightarrow 0$, ($\text{sn}, \text{cn}, \text{dn}$) $\rightarrow$ ($\sin, \cos, 1$).

2.2 Methodology

For a nonlinear PDE system $P(u, v, u_t, v_t, u_x, v_x, \dots) = 0$:

Step 1 (traveling-wave transformation). Set $\xi = x - ct$, $u(x, t) = u(\xi)$, $v(x, t) = v(\xi)$.

Step 2 (auxiliary ODE). Expand the solution in an auxiliary function $F(\xi)$ satisfying the quartic first-order equation (Ebaid and Aly, 2012)

$$(F'(\xi))^2 = l_2 F^4(\xi) + l_1 F^2(\xi) + l_0, \quad (3)$$

with l_0, l_1, l_2 fixed by the chosen Jacobi function; locally $F' = \sigma \sqrt{l_2 F^4 + l_1 F^2 + l_0}$, $\sigma = \pm 1$.

Step 3 (balanced rational Ansatz). Homogeneous balance between the highest derivative and the dominant nonlinear term gives

$$u(\xi) = g_0 + g_1 F(\xi) + \frac{f_1}{F(\xi)}. \quad (4)$$

Step 4 (algebraic system). Substituting (4) and (3) into the reduced ODE and equating coefficients of independent powers of F yields an overdetermined algebraic system for g_0, g_1, f_1 and the physical parameters.

3 Reduction of the Coupled WBK System

The coupled WBK shallow-water system is

$$u_t + u u_x + v_x + b u_{xx} = 0, \quad (5)$$

$$v_t + (uv)_x + a u_{xxx} - b v_{xx} = 0, \quad (6)$$

with $a, b \in \mathbb{R}$ parametrizing dispersion. With $\xi = x - ct$, and assuming $c > 0$ without loss of generality (the case $c < 0$ follows from the space-inversion symmetry $\xi \rightarrow -\xi$), the system becomes

$$-c u' + u u' + v' + b u'' = 0, \quad (7)$$

$$-c v' + (uv)' + a u''' - b v'' = 0. \quad (8)$$

Integrating the first equation once gives $v = c u - \frac{1}{2}u^2 - b u' + C_1$. Substituting into the second and integrating once more gives the decoupled second-order ODE

$$(b^2 + a)u'' + \frac{3}{2}cu^2 - \frac{1}{2}u^3 - c^2u + C_2 = 0. \quad (9)$$

Remark 1 (Nature of the specialization $C_1 = C_2 = 0$). Setting $C_1 = C_2 = 0$ in (9) is the zero-background integration-constant subcase: it renders the reduced ODE (11) autonomous and free of an additive forcing term – the minimal condition needed for the ansatz (4) to close algebraically at the order at which it is posed. We do not claim this subcase corresponds to a vanishing periodic mean of u or v ; that mean is not computed in this paper (see also Remark 10 below for the analogous point when $C_1 \neq 0$). It should *not* be read as a boundary condition satisfied pointwise by the periodic profiles constructed below. The non-trivial rational solutions possess isolated singularity lattices where $F(\xi) = 0$, defining regular classical solutions on pole-truncated subdomains; only the degenerate branches ($f_1 = 0$ or $g_1 = 0$, Remark 3) vanish asymptotically as $|\xi| \rightarrow \infty$. The pair (C_1, C_2) are free integration constants, not constants fixed by the physics of (5)–(6); other choices remain available in principle and would require re-solving the resulting inhomogeneous algebraic system. This is left for future work.

With $C_1 = C_2 = 0$,

$$v(\xi) = c u(\xi) - \frac{1}{2}u^2(\xi) - b u'(\xi), \quad (10)$$

$$(b^2 + a)u''(\xi) + \frac{3}{2}cu^2(\xi) - \frac{1}{2}u^3(\xi) - c^2u(\xi) = 0. \quad (11)$$

4 Solution of the Algebraic System

Substituting (4) together with (3) into (11) and equating coefficients of $F^{\pm k}$ gives the overdetermined system

$$\begin{aligned} 3cf_1^2 - 3f_1^2g_0 &= 0, & 4(b^2 + a)l_0f_1 - f_1^3 &= 0, \\ -2c^2g_0 + 6cf_1g_1 + 3cg_0^2 - 6f_1g_0g_1 - g_0^3 &= 0, \\ 2(b^2 + a)l_1g_1 - 2c^2g_1 + 6cg_0g_1 - 3f_1g_1^2 - 3g_0^2g_1 &= 0, \\ 2(b^2 + a)l_1f_1 - 2c^2f_1 + 6cf_1g_0 - 3f_1^2g_1 - 3f_1g_0^2 &= 0, \\ 3cg_1^2 - 3g_0g_1^2 &= 0, & 4(b^2 + a)l_2g_1 - g_1^3 &= 0. \end{aligned} \quad (12)$$

For non-trivial amplitudes ($f_1g_1 \neq 0$) and $c > 0$, the principal-sign branch of the solution is

$$g_0 = c, \quad g_1^2 = 4(b^2 + a)l_2, \quad f_1^2 = 4(b^2 + a)l_0, \quad (13)$$

subject to the compatibility condition

$$(b^2 + a)(12\sqrt{l_0l_2} - 2l_1) = c^2 \Rightarrow a + b^2 = \frac{c^2}{2D}, \quad D := 6\sqrt{l_0l_2} - l_1, \quad (14)$$

where $\sqrt{l_0 l_2}$ denotes the principal square root. Consequently, taking the principal (non-negative) square root of each amplitude,

$$g_1 = c \sqrt{\frac{2l_2}{D}}, \quad f_1 = c \sqrt{\frac{2l_0}{D}}, \quad (15)$$

This fixes the magnitude of each amplitude only; their relative sign is a separate question, addressed immediately below in Remark 3 and, for the $C_1 \neq 0$ generalization, via the explicit sign factor $\varepsilon = \text{sign}(D)$ in Theorem 1's Eq. (36). And

$$u'(\xi) = \left(g_1 - \frac{f_1}{F^2(\xi)} \right) F'(\xi), \quad F'(\xi) = \sigma \sqrt{l_2 F^4(\xi) + l_1 F^2(\xi) + l_0}. \quad (16)$$

Equation (16) is a local identity, valid at points where $F(\xi) \neq 0$; zeros of F produce poles of u (through f_1/F) and of u' , excluded from the pole-truncated domains used throughout (Remark 6). As just noted, (15) gives the correct relative sign exactly when $D > 0$ (Remark 3). All families in Table 1 have $D > 0$ throughout their stated real domain except Case 4, for which the $D < 0$ branch requires a correlated sign correction; see Remark 5.

Remark 2 (Notation). Equation (15) is used throughout in this single, unambiguous closed-radical form, obtained directly from (13) and (14) by substitution. Every occurrence of g_1, f_1 below is computed from (15); Appendix B gives a numerical cross-check for Case 1.

Remark 3 (Degenerate branches, and the sign of $f_1 g_1$). From (13), $f_1 g_1 = \pm 4(b^2 + a)\sqrt{l_0 l_2}$, and the coefficient of F^2 in (12) requires the specific value $3f_1 g_1 = 2(b^2 + a)l_1 + c^2$. Writing $l_1 = 6\sqrt{l_0 l_2} - D$ from (14) and using $a + b^2 = c^2/(2D)$, this required value equals $6c^2\sqrt{l_0 l_2}/D$, whereas the principal (both non-negative) choice (15) gives $3f_1 g_1 = 6c^2\sqrt{l_0 l_2}/|D|$. These agree precisely when $D > 0$; when $D < 0$ they have opposite sign, so the compatible branch instead takes $f_1 \mapsto -f_1$ (equivalently $g_1 \mapsto -g_1$) in (15). This is not merely a formal remark: for real F and $D < 0$, having l_0, l_2 both negative (so that $l_0/D, l_2/D > 0$ and g_1, f_1 remain real) occurs among the tabulated cases only for Case 4, which is why Remark 5 is needed specifically there. The algebraic system (12) also admits degenerate solitary branches when $f_1 = 0$ (admissible if $l_0 = 0$) or $g_1 = 0$ (if $l_2 = 0$), corresponding to the single-term hyperbolic kink profiles $u = c \pm \text{ctanh}(x - ct)$ isolated by [Xie et al. \(2001\)](#). These degenerate branches require $l_0 = 0$ or $l_2 = 0$ and are therefore *not* the $m \rightarrow 1$ limit of the (generically $l_0, l_2 \neq 0$) Cases constructed below (Table 1); see Section 7.

5 The Ten Independent Solution Families

With $\xi = x - ct$, $m \in (0, 1)$, and the auxiliary data in Table 1, every family has the closed form

$$u_i(\xi) = c + g_{1,i}F_i(\xi) + \frac{f_{1,i}}{F_i(\xi)}, \quad v_i(\xi) = c u_i(\xi) - \frac{1}{2}u_i^2(\xi) - b u_i'(\xi), \quad (17)$$

with $g_{1,i}, f_{1,i}$ from (15) and u_i' from (16).

The raw expansion algorithm of Section 4 in fact produces fifteen candidate representations, but five of them are algebraically identical, by standard Jacobi reciprocal identities, to representations already produced by a different choice of F : applying the algorithm with $F = \text{ns}, \text{dc}, \text{nc}, \text{nd}$, or $\text{sn}/(1 + \text{cn})$ gives, after simplification, exactly the same function $u(\xi)$ as $F = \text{sn}, \text{cd}, \text{cn}, \text{dn}$, or $\text{ns} + \text{cs}$, respectively. These five duplicate representations are omitted from Table 1, which lists only the ten representations that are pairwise algebraically independent (Table 2 records the omissions for traceability). Eliminating the duplicates at this stage – rather than tabulating fifteen entries and noting the identities afterward – is possible precisely because the five omitted forms are related to already-listed ones by an explicit change of F alone, with g_0, g_1, f_1 unchanged; no such elimination is available among the remaining ten.

Table 1: Auxiliary data for the ten algebraically independent families. $D = 6\sqrt{l_0 l_2} - l_1$. R = real-valued family, C = complex-valued formal branch.

Case	$F(\xi)$	l_0	l_1	l_2	$F'(\xi)$	D	Char.
1	sn	1	$-(1 + m^2)$	m^2	cn dn	$m^2 + 6m + 1$	R
2	cd	1	$-(1 + m^2)$	m^2	$-(1 - m^2)\text{sn}/\text{dn}^2$	$m^2 + 6m + 1$	R
3	cn	$1 - m^2$	$2m^2 - 1$	$-m^2$	$-\text{sn dn}$	$6im\sqrt{1 - m^2} - (2m^2 - 1)$	C
4	dn	$m^2 - 1$	$2 - m^2$	-1	$-m^2 \text{sn cn}$	$6\sqrt{1 - m^2} - (2 - m^2)$	C/R [†]
5	sc	1	$2 - m^2$	$1 - m^2$	dc nc	$6\sqrt{1 - m^2} - (2 - m^2)$	R*
6	sd	1	$2m^2 - 1$	$-m^2(1 - m^2)$	cd nd	$6im\sqrt{1 - m^2} - (2m^2 - 1)$	C
7	ns + cs	1/4	$(1 - 2m^2)/2$	1/4	$-\text{ds } F$	$1 + m^2$	R
8	sn + i cn	$m^2/4$	$(m^2 - 2)/2$	$m^2/4$	$-i \text{ dn } F$	$1 + m^2$	C
9	cn/(1 + sn)	$(1 - m^2)/4$	$(1 + m^2)/2$	$(1 - m^2)/4$	$-\text{dn}/(1 + \text{sn})$	$1 - 2m^2$	R**

Case	$F(\xi)$	l_0	l_1	l_2	$F'(\xi)$	D	Char.
10	$m \operatorname{sn} + i \operatorname{dn}$	$1/4$	$(1 - 2m^2)/2$	$1/4$	$m \operatorname{cn}(\operatorname{dn} - im \operatorname{sn})$	$1 + m^2$	C

Real for $m < m^ \approx 0.98517$, the positive root of $m^4 + 32m^2 - 32 = 0$.

**Real for $m < 1/\sqrt{2}$ (guaranteeing $D > 0$).

†Complex for $m < m^*$; real, via the sign-corrected branch (23), for $m^* < m < 1$ (Remark 5).

Table 2: Representations omitted from Table 1 as algebraic duplicates (retained for traceability only).

Omitted F	Algebraic relation	Identical to	Case retained
ns	$u(\operatorname{ns}) \equiv u(\operatorname{sn})$	Case 1	1
dc	$u(\operatorname{dc}) \equiv u(\operatorname{cd})$	Case 2	2
nc	$u(\operatorname{nc}) \equiv u(\operatorname{cn})$	Case 3	3
nd	$u(\operatorname{nd}) \equiv u(\operatorname{dn})$	Case 4	4
$\operatorname{sn}/(1 + \operatorname{cn})$	$u(\cdot) \equiv u(\operatorname{ns} + \operatorname{cs})$	Case 7	7

Cases 1–2 and 3–4 share the same D -type expression (up to the sign inside the square root); explicit closed forms for representative cases are:

Case 1 (real). $D = m^2 + 6m + 1$, $g_1 = c\sqrt{2m^2/D}$, $f_1 = c\sqrt{2/D}$, $F = \operatorname{sn}(\xi, m)$:

$$u_1 = c + c\sqrt{\frac{2}{D}}(m \operatorname{sn} + \operatorname{sn}^{-1}), \quad u'_1 = c\sqrt{\frac{2}{D}}\left(m - \frac{1}{\operatorname{sn}^2}\right) \operatorname{cn} \operatorname{dn}. \quad (18)$$

Case 2 (real). Same D , g_1 , f_1 as Case 1, $F = \operatorname{cd}(\xi, m)$:

$$u_2 = c + c\sqrt{\frac{2}{D}}(m \operatorname{cd} + \operatorname{dc}). \quad (19)$$

Case 7 (real). $D = 1 + m^2$, $g_1 = f_1 = c/\sqrt{2(1 + m^2)}$, $F = \operatorname{ns} + \operatorname{cs}$:

$$u_7 = c + \frac{c}{\sqrt{2(1 + m^2)}}\left(F + \frac{1}{F}\right). \quad (20)$$

This is algebraically identical to the equivalent form obtained from $F = \operatorname{sn}/(1 + \operatorname{cn})$ (Table 2); we use this equivalent form in Figure 3 and Appendix A below, since it makes the half-argument structure of Remark 8 transparent.

Case 9 (real**). $D = 1 - 2m^2 > 0$, $g_1 = f_1 = c\sqrt{\frac{1-m^2}{2(1-2m^2)}}$, $F = \operatorname{cn}/(1 + \operatorname{sn})$:

$$u_9 = c + g_1\left(F + \frac{1}{F}\right). \quad (21)$$

Case 10 (complex). The non-real branches do not represent observable water-surface elevations but are exact formal solutions of the reduced ODE. For $F_{10} = m \operatorname{sn} + i \operatorname{dn}$, $D = 1 + m^2$, $g_1 = f_1 = c/\sqrt{2(1+m^2)}$:

$$u_{10} = c + \frac{c}{\sqrt{2(1+m^2)}} \left(F_{10} + \frac{1}{F_{10}} \right). \quad (22)$$

The remaining cases (3, 4, 5, 6, 8) follow the same pattern as in Table 1; full verification of the composite cases is given in Appendix A.

Remark 4 (Real versus complex-valued families). On the principal branch of (15), a family is real-valued if $l_0 \geq 0$, $l_2 \geq 0$, $D > 0$ (so $g_1, f_1 \in \mathbb{R}$ there) and $F(\xi)$ is real; this criterion is sufficient but not globally necessary, since Case 4 additionally has a real, sign-corrected branch for $D < 0$ (Remark 5) that the principal-branch criterion alone does not capture. The five families satisfying this throughout $0 < m < 1$ (or the stated sub-range) are 1,2,5,7,9 (Case 5 restricted to $m < m^* \approx 0.98517$, Case 9 to $m < 1/\sqrt{2}$). Cases 3, 6 have $l_0 l_2 < 0$, so D itself is non-real, and these cases are complex for every $0 < m < 1$; Cases 8, 10 involve an explicitly complex auxiliary function F and are likewise complex for every $0 < m < 1$. These four (3,6,8,10) are strictly complex formal solutions. Case 4 has $l_0 < 0$ and $l_2 < 0$ together: for $0 < m < m^* \approx 0.98517$ it has real $D > 0$ but imaginary amplitudes on the principal branch (since $l_2/D < 0$ there), so it is complex on that range; but because l_0, l_2 are *both* negative, the sign-corrected branch of Remark 3 makes it real, not complex, on the complementary range $m^* < m < 1$ where $D < 0$ – see Remark 5. Case 4 is therefore not strictly complex for all $0 < m < 1$, unlike Cases 3,6,8,10.

Remark 5 (A real branch of Case 4 for $m^* < m < 1$). For Case 4 ($F = \operatorname{dn}$), $l_0 = m^2 - 1$, $l_2 = -1$, and $D = 6\sqrt{1-m^2} - (2-m^2)$, so $D = 0$ exactly at the same $m^* \approx 0.98517$ (the positive root of $m^4 + 32m^2 - 32 = 0$) as Case 5, since both share the identical expression for D . For $0 < m < m^*$, $D > 0$ while $l_0, l_2 < 0$, so $l_0/D, l_2/D < 0$: the principal branch (15) gives imaginary g_1, f_1 , consistent with the classification of Case 4 as complex on this range. For $m^* < m < 1$, however, $D < 0$ while $l_0, l_2 < 0$ still, so $l_0/D, l_2/D > 0$: *both* amplitude magnitudes become real. By Remark 3, the compatible branch on this range takes the principal (non-negative) value for one amplitude and the negative value for the other, e.g.

$$g_1 = c \sqrt{\frac{2l_2}{D}} (> 0), \quad f_1 = -c \sqrt{\frac{2l_0}{D}} (< 0), \quad m^* < m < 1, \quad (23)$$

(the overall choice of which amplitude carries the negative sign is a labeling convention only, since $u = c + g_1 F + f_1/F$ is unchanged under $(g_1, f_1, F) \mapsto (-g_1, -f_1, -F)$ and $\operatorname{dn}(\xi, m)$ is even). This was verified independently, both symbolically (substitution into (12)) and to 50-digit numerical precision against the *original*, unintegrated PDEs (5)–(6)

(not merely the reduced ODE), at $m = 0.99 > m^*$, $c = 1$, $b = 0.6$: this gives $D \approx -0.17350$, $g_1 \approx 3.39524$, $f_1 \approx -0.47896$, $a + b^2 = c^2/(2D) \approx -2.88191$ so $a \approx -3.24191$, with both original PDE residuals below 10^{-49} at a sample point. Taking instead *both* amplitudes with the principal non-negative sign on this same range gives a genuine, reproducible discrepancy in the second (unintegrated) PDE at the level of order unity, confirming that the relative sign in (23) – not the principal branch of (15) – is the one required by (5)–(6) on $m^* < m < 1$. This real branch of Case 4 is new relative to the classification in earlier drafts of this material, which stated Case 4 as complex for all $0 < m < 1$.

Remark 6 (Regularity on a pole-truncated domain). For a real-valued classical profile on a pole-truncated domain, we require: (i) $0 < m < 1$; (ii) $D > 0$ as in Table 1, or, for Case 4 on $m^* < m < 1$, $D < 0$ together with the sign-corrected branch (23); (iii) a is fixed via (14) once (c, b, m) are chosen; (iv) isolated zeros of $F(\xi)$ are excluded from the domain, since f_1/F has a pole there; and (v) any point where $F(\xi)$ itself is undefined – either because F is a ratio with a vanishing denominator (e.g. $F = ns + cs$ at $sn = 0$, or $F = cn/(1 + sn)$ at $sn = -1$) or because F itself has poles as one of the nine reciprocal Jacobi functions – is likewise excluded. This establishes regularity of the profile on the stated domain; it is not a claim of physical admissibility in every sense (e.g., boundedness of energy or realizability as an initial-value problem), which is not addressed here.

6 Taxonomy, Redundancies, and Novelty Relative to Prior Literature

6.1 Algebraic redundancies

The five representations omitted from Table 1 were identified, by the standard Jacobi reciprocal identities used throughout, as algebraically identical to five of the ten listed representations; the precise correspondence is given in Table 2 above. No further identities of this kind hold among the ten representations that remain: each is algebraically independent of the other nine under the parameterization g_0, g_1, f_1 of (15), in the sense that no two are related by a substitution of F alone with g_0, g_1, f_1 held fixed. Table 2's "Case retained" column records this correspondence; five of the ten retained cases are real-valued throughout their range, and a sixth (Case 4) is real on part of its range (Remark 4).

6.2 Structural independence of the composite families

The elliptic equation rational expansion method of [Chen et al. \(2005\)](#) is, like the present one, built on a quartic auxiliary equation $\phi'^2 = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4$, and its enumeration of admissible auxiliary functions already lists the same composite combinations used here – $\phi = ns \pm cs$, $sn \pm i cn$, $nc \pm sc$ – with h_0, h_2, h_4 numerically identical to l_0, l_1, l_2 of Cases 7, 8, 9 here. Sharing the auxiliary function is therefore not enough to establish novelty; independence must be shown at the level of the final velocity ansatz. For the balance $m_u = 1$ required by the WBK system, the velocity-field ansatz of [Chen et al. \(2005\)](#) (their Eq. (2.4)) specializes to

$$U(\eta) = a_0 + a_1\phi + \frac{b_1\phi'}{l\phi + 1}, \quad (24)$$

i.e., a sum of nonnegative powers of ϕ (and of ϕ') – no term of the form $1/\phi$ appears in the definition of their method. After eliminating ϕ' via the quartic auxiliary equation and their compatibility system, every published closed-form solution of [Chen et al. \(2005\)](#) reduces, for each explicit case in their Table, to a bounded, Möbius-type (fractional-linear) function of ϕ ,

$$U = a_0 + \frac{N\phi}{D\phi + 1} \quad (25)$$

for constants N, D depending on $l, h_0, \dots, h_4$ and the wave speed. By contrast, the ansatz (4) used here is $u = g_0 + g_1F + f_1/F$, which explicitly contains the reciprocal term f_1/F .

Lemma 1 (Structural independence). For $f_1 \neq 0$ and $g_1 \neq 0$, the ansatz (4) cannot be written in the Möbius form (25) as a function of the same auxiliary variable F . Consequently, within the velocity-field ansatz and parameterization of [Chen et al. \(2005\)](#) specified above, the composite families u_7, u_8, u_9, u_{10} are not represented by their construction, including for their families built from the identical composite auxiliary functions.

Proof. Write (4) as a single rational function of F : $u = \frac{g_1F^2 + g_0F + f_1}{F}$, a ratio of a degree-2 numerator to a degree-1 denominator (in lowest terms, since $f_1 \neq 0$ means $F = 0$ is a genuine pole not cancelled by the numerator). Any Möbius function (25) is, by definition, a ratio of a degree-1 numerator to a degree-1 denominator (plus the additive constant a_0 , itself absorbable into a degree-1/degree-1 form whenever $D \neq 0$). A degree-(2,1) rational function of F can equal a degree-(1,1) rational function of the same variable only if the degree-2 coefficient vanishes, i.e., $g_1 = 0$ – contradicting the hypothesis. Equivalently: as $F \rightarrow \infty$ along any path avoiding the poles, $u \rightarrow g_0 + g_1F \rightarrow \infty$, whereas every Möbius function (25) satisfies $U \rightarrow a_0 + N/D$, a finite limit. An unbounded function of a variable cannot coincide with a bounded function of the same variable. Since $g_1, f_1 \neq 0$ in each of Cases 7,8,9,10 (Table 1), this establishes structural independence for all four.

Remark 7 (Scope of the independence claim). Lemma 1 is established for the velocity-field ansatz $U = a_0 + a_1\phi + b_1\phi'/(l\phi + 1)$ exactly as specialized by [Chen et al. \(2005\)](#) for the balance $m_u = 1$, and for the traveling-wave parameterization used throughout this paper. Accordingly, the novelty claim is restricted to that stated ansatz and parameterization: the argument shows that no solution reachable from (25) coincides, as a function of the *same* auxiliary variable $F = \phi$, with a family from Table 1. It does not, by itself, rule out equivalence under an unstated reparametrization of the auxiliary variable (e.g., $\phi \mapsto 1/\psi$), since such a substitution can convert a bounded Möbius function of ϕ

into an unbounded function of ψ ; [Chen et al. \(2005\)](#) do not introduce such a reparametrization, and none is applied here.

Remark 8 (Supporting geometric observation). Independently of the boundedness argument, the composite auxiliary functions of Cases 7,8,9,10 carry an intrinsic half-argument structure absent from a single Jacobi function evaluated at ξ . We verify this explicitly for the equivalent form of Case 7 used in Figure 3 (Table 2). Write $u = \xi/2$ and $s = \text{sn}(u, m)$, $c_u = \text{cn}(u, m)$, $d_u = \text{dn}(u, m)$. The standard Jacobi double-argument formulas give

$$\text{sn}(\xi, m) = \frac{2sc_u d_u}{1 - m^2 s^4}, \quad \text{cn}(\xi, m) = \frac{1 - 2s^2 + m^2 s^4}{1 - m^2 s^4}, \quad (26)$$

so that, using $c_u^2 = 1 - s^2$,

$$1 + \text{cn}(\xi, m) = \frac{(1 - m^2 s^4) + (1 - 2s^2 + m^2 s^4)}{1 - m^2 s^4} = \frac{2(1 - s^2)}{1 - m^2 s^4} = \frac{2c_u^2}{1 - m^2 s^4}. \quad (27)$$

Dividing gives the exact half-argument identity

$$\frac{\text{sn}(\xi, m)}{1 + \text{cn}(\xi, m)} = \frac{2sc_u d_u / (1 - m^2 s^4)}{2c_u^2 / (1 - m^2 s^4)} = \frac{sd_u}{c_u} = \frac{\text{sn}(\xi/2, m) \text{dn}(\xi/2, m)}{\text{cn}(\xi/2, m)}, \quad (28)$$

which we write out in full (rather than via the shorthand sc) to avoid any ambiguity between the product $sc(\xi/2, m) \text{dn}(\xi/2, m)$ (equal to (28), since $sc = \text{sn}/\text{cn}$ by definition (2)) and the superficially similar quotient $sc(\xi/2, m)/\text{dn}(\xi/2, m)$, which is a different function and does *not* equal the left-hand side. Identity (28) gives $F(\xi)$ an explicit half-argument representation that a linear or simple rational function of $\text{sn}(\xi, m)$, $\text{cn}(\xi, m)$, or $\text{dn}(\xi, m)$ at the same argument ξ cannot reproduce; the same holds for Cases 8, 9, 10. (We do not assert a period-doubling statement here, which would require separately establishing a period relation, not merely the pointwise identity (28).) This is a useful geometric characterization but, unlike Lemma 1, it is specific to which Jacobi functions enter the composite and does not by itself establish independence from [Chen et al. \(2005\)](#).

6.3 A universal Weierstrass-elliptic representation

This representation is included to give an alternative elliptic-curve form of the traveling-wave solutions, in which the dependence on the associated cubic invariants g_2, g_3 – and hence on the discriminant $\Delta = g_2^3 - 27g_3^2$ that controls the real/complex character of the underlying elliptic curve – is made explicit; it is algebraically equivalent to the corresponding Jacobi-elliptic form under the substitution $\wp = l_2 F^2 + l_1/3$ below, not an independent family of solutions. We use the standard normalization $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ (Whittaker–Watson convention, as in [Ebaid and Aly \(2012\)](#)); other sources scale g_2, g_3 differently (e.g. by absorbing the factor 4), so (31) below should be read together with this stated convention. The auxiliary equation (3) contains only even powers of F . This biquadratic structure means that every family in Table 1 – not just a specially chosen

subset – admits an equivalent Weierstrass-elliptic representation, obtained by a single, uniform change of variable, independently of which Jacobi function F happens to be.

Proposition 1. For any l_0, l_1, l_2 , define

$$\wp(\xi) := l_2 F^2(\xi) + \frac{l_1}{3}. \quad (29)$$

If F satisfies (3), then $\wp$ satisfies the Weierstrass equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - g_2\wp(\xi) - g_3, \quad (30)$$

with invariants

$$g_2 = \frac{4}{3}(l_1^2 - 3l_0l_2), \quad g_3 = \frac{4}{27}l_1(9l_0l_2 - 2l_1^2). \quad (31)$$

Consequently, for $D \neq 0$ and away from the zeros of F (equivalently, $\wp(\xi) \neq l_1/3$), every family (17) has the equivalent closed form

$$u(\xi) = c + c \sqrt{\frac{2(\wp(\xi) - l_1/3)}{D}} + c \sqrt{\frac{2l_0l_2}{D(\wp(\xi) - l_1/3)}}, \quad (32)$$

where $\wp = \wp(\xi; g_2, g_3)$ is the Weierstrass elliptic function with invariants (31).

Proof. Differentiating (29) gives $\wp' = 2l_2FF'$, so $(\wp')^2 = 4l_2^2F^2(F')^2 = 4l_2^2F^2(l_2F^4 + l_1F^2 + l_0)$ by (3). Substituting $F^2 = (\wp - l_1/3)/l_2$ from (29) and expanding gives, after simplification, $(\wp')^2 = 4\wp^3 - \frac{4}{3}(l_1^2 - 3l_0l_2)\wp - \frac{4}{27}l_1(9l_0l_2 - 2l_1^2)$, which is (30)–(31); this identity was also verified independently by direct symbolic expansion (Appendix C). Equation (32) follows from (17) and (15) by substituting $F = \sqrt{(\wp - l_1/3)/l_2}$. ■

Remark 9 (On a further attempted extension). A further attempted extension of this material sought a comparable Weierstrass-form result for Cases 7,8,9,10 by asserting, for a base family, invariants of the same form as (31) (there written with $P = l_2$, $Q = l_1$, $R = l_0$) – which indeed agree with Proposition 1 once translated – but then extended this into five further “branches” (denoted W2–W6 in that attempt) containing quantities such as α , m_1 , ω , and D_w that are nowhere defined in terms of l_0, l_1, l_2, c or D , and that do not reduce, upon substitution, to solutions of the algebraic system (12). Those branches could not be verified and are not reproduced here. Proposition 1 is the complete, verified content that can honestly be extracted from that attempt: a single universal reduction, applicable to every family already in Table 1, requiring no new undetermined parameters.

Table 3: Comparison with representative prior constructions for the WBK system

Reference	Auxiliary equation / method	Solution types; overlap
Xie et al. (2001)	Hyperbolic ansatz	Solitons; the degenerate branch $f_1 = 0$ of (12)
El-Sabbagh and El-	First-integral method (Division Theorem)	Standard hyperbolic/trigonometric waveforms; no composite elliptic structures

Reference	Auxiliary equation / method	Solution types; overlap
Ganaini (2012)		
Chen et al. (2005)	Elliptic equation rational expansion method (quartic auxiliary $h_0, \dots, h_4$); ansatz restricted to nonnegative powers of ϕ (Möbius-bounded)	Includes composite $\phi = ns \pm cs, sn \pm i cn, nc \pm sc$, but only via the bounded ansatz (25); distinct from the present unbounded ansatz by Lemma 1
Wang et al. (2020)	Generalized Riccati mapping	Solitons and periodic waves; different ansatz
Xu et al. (2017)	Wronskian / bilinear method	Soliton interaction profiles; different solution class
Zhang et al. (2008)	Lie symmetry reduction	Invariant similarity waves; different reduction
Present work	$(F')^2 = l_2 F^4 + l_1 F^2 + l_0$	Composite rational families 7,8,9,10; new by Lemma 1

7 Physical Interpretation, Parameter Domains, and Visualizations

All profiles below are computed directly from (17), with $dn^2 = 1 - m^2 sn^2$, on a domain that excludes a symmetric band around each pole of $F(\xi)$.

Figure 1 (near-trigonometric branch, $m \rightarrow 0$). As $m \rightarrow 0$, Case 1 approaches the singular periodic train $u_1 \rightarrow c + \sqrt{2} c \csc(x - ct)$. Figure 1 shows the pole-truncated profile at $m = 0.05$, $c = 1.5$.

Figure 2 (corrected solitary limit, $m \rightarrow 1$). *This corrects an error in an earlier treatment of this material.* It is tempting to expect Case 1 to approach the classical smooth kink $u \rightarrow c + c \tanh(x - ct)$ of [Xie et al. \(2001\)](#) as $m \rightarrow 1$, since $sn(\xi, m) \rightarrow \tanh \xi$. However, the reciprocal term does not vanish: $sn^{-1}(\xi, m) \rightarrow \coth \xi$, which diverges at $\xi = 0$ (and at every zero of sn). More generally, for any family in Table 1 built from a reciprocal Jacobi function (ns, nc, nd, cs, ds, dc , or a composite containing one), the $m \rightarrow 1$ limit is singular on the domain $\xi \in \mathbb{R}$ at the excluded points where the corresponding direct function (sn, cn, dn , respectively) vanishes; only the degenerate branches of Remark 3 ($f_1 = 0$ or $g_1 = 0$) are regular on all of $\mathbb{R}$ in this limit. Concretely, at $m \rightarrow 1$, $D \rightarrow 8$ and $c\sqrt{2/D} \rightarrow c/2$, so

$$u_1 \rightarrow c + \frac{c}{2}(\tanh \xi + \coth \xi), \quad (33)$$

which is singular at $\xi = 0$ for every finite c . Figure 2 plots this at $m = 0.999$, $c = 1.5$, together with the genuinely smooth kink $c + c \tanh \xi$: the two curves agree away from the pole, but Case 1 diverges at $\xi = 0$ where the smooth kink does not. The smooth kink is not the $m \rightarrow 1$ limit of Case 1; it is the distinct degenerate branch $f_1 = 0$ (equivalently $l_0 = 0$)

of the algebraic system (12) (Remark 3), which lies outside the generic ($l_0, l_2 \neq 0$) family tabulated in Table 1.

Figure 3 (composite rational train). Case 7, shown via its equivalent form $F = sn/(1 + cn)$ (Table 2), with $m = 0.5, c = 1.5$: a periodic train with the half-argument, multi-crest structure discussed above.

Figure 4 (localized trough packet). Case 2 with $F = cd, m = 0.85, c = 1.4$.

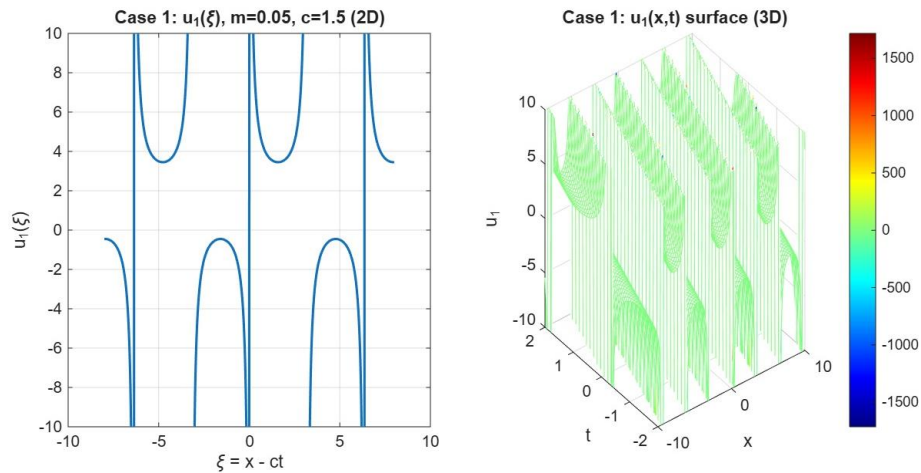


Figure 1: Case 1 near-trigonometric branch, $m = 0.05, c = 1.5$. Left: profile $u_1(\xi)$ vs. $\xi = x - ct$. Right: the same family shown as a 3D surface $u_1(x, t)$, making explicit that the poles of Figure 1 (left) are the traces, moving at speed c , of the pole lattice visible here.

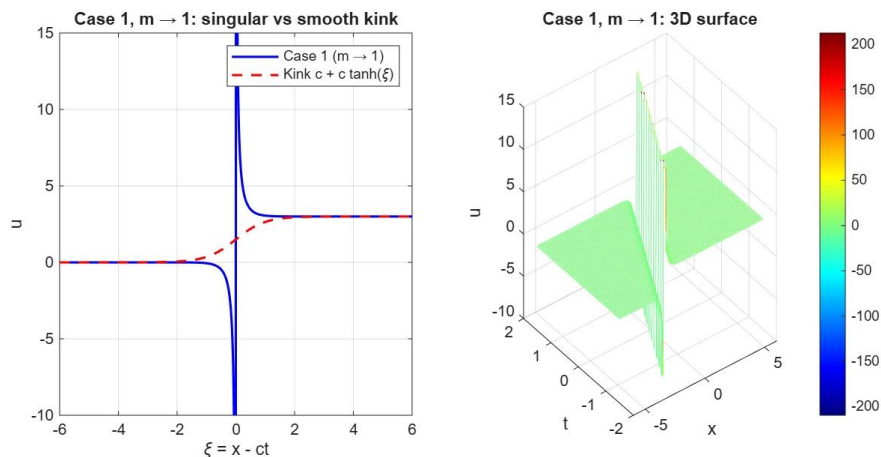


Figure 2: Case 1 as $m \rightarrow 1$ ($m = 0.999, c = 1.5$). Left: profile (solid) compared with the distinct smooth-kink degenerate branch $c + c \tanh \xi$ (dashed); the family remains singular, only the degenerate branch is smooth. Right: the corresponding $u(x, t)$ surface, showing the singularity as a ridge running along the characteristic line $x = ct$.

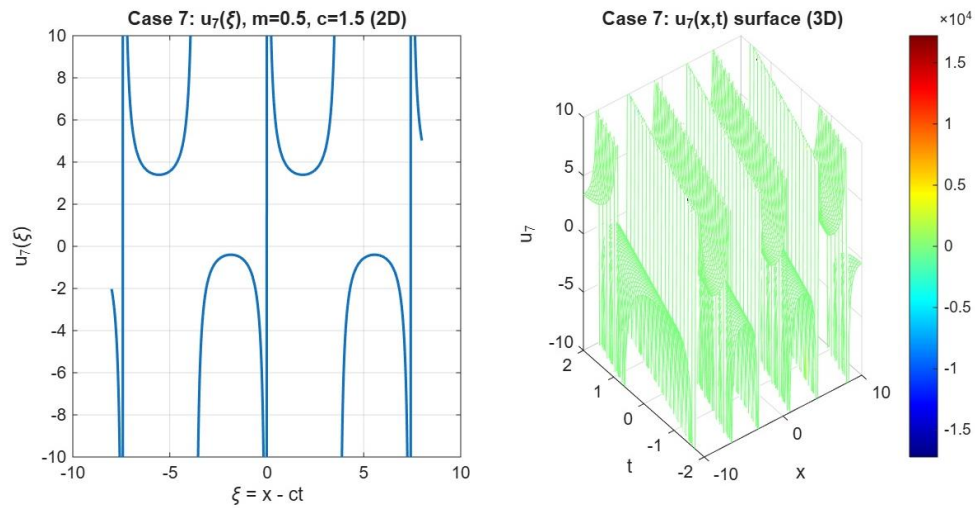


Figure 3: Case 7 (equivalent form $F = sn/(1 + cn)$), composite rational periodic train, $m = 0.5$, $c = 1.5$. Left: profile $u_7(\xi)$. Right: $u_7(x, t)$ surface, showing the half-argument multi-crest structure of Remark 8 propagating at a speed of c .

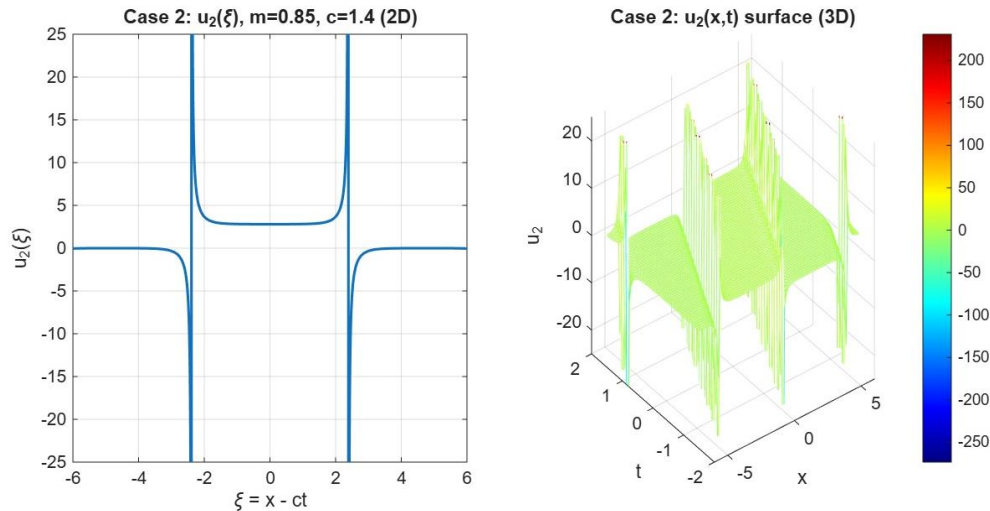


Figure 4: Case 2, localized trough packet, $m = 0.85$, $c = 1.4$. Left: profile $u_2(\xi)$. Right: the corresponding $u_2(x, t)$ surface.

8 Beyond a Vanishing Background Level: The General C_1 Family

Remark 1 left the case $C_1, C_2 \neq 0$ – steady waves with a *nonzero* background free-surface level – as a direction for future work, since it requires re-solving the algebraic system with the additive forcing term reinstated. We carry this out here in closed form. Integrating (5)–(6) with C_1, C_2 retained gives, in place of (10)–(11),

$$v(\xi) = c u(\xi) - \frac{1}{2} u^2(\xi) - b u'(\xi) + C_1, \quad (34)$$

$$(b^2 + a)u''(\xi) + \frac{3}{2}cu^2(\xi) - \frac{1}{2}u^3(\xi) + (C_1 - c^2)u(\xi) + C_2 = 0. \quad (35)$$

Theorem 1 (General C_1 family). Assume $l_0 l_2 \geq 0$ (so that $\sqrt{l_0 l_2}$ is real, as already required in Table 1), $D := 6\sqrt{l_0 l_2} - l_1 \neq 0$, and $c^2 + 2C_1 > 0$ (strictly; at equality $\tilde{c} = 0$, forcing $g_1 = f_1 = 0$ by (36) below, which contradicts the non-trivial-amplitude hypothesis and must instead be treated as a degenerate branch, as in Remark 3). Substituting the ansatz (4) into (35) and equating coefficients of $F^{\pm k}$ gives, for non-trivial amplitudes ($f_1, g_1 \neq 0$) and $c > 0$, the solution

$$g_0 = c, \quad C_2 = -cC_1, \quad g_1 = \tilde{c} \sqrt{\frac{2l_2}{D}}, \quad f_1 = \varepsilon \tilde{c} \sqrt{\frac{2l_0}{D}}, \quad \varepsilon := \text{sign}(D) \quad (36)$$

$$= \frac{D}{|D|},$$

where D is unchanged from (14) and

$$\tilde{c} := \sqrt{c^2 + 2C_1} (> 0), \quad a + b^2 = \frac{\tilde{c}^2}{2D} = \frac{c^2 + 2C_1}{2D}. \quad (37)$$

That is, g_1 always takes the principal (non-negative) square root, while f_1 takes the principal root only when $D > 0$ and its negative when $D < 0$; this is the exact analogue, for general C_1 , of the sign correction already established for $C_1 = 0$ in Remark 3, and it reduces to it (with $\varepsilon = +1$) whenever $D > 0$, which covers every tabulated case except Case 4 on $m^* < m < 1$ (Remark 5, now extended to $C_1 \neq 0$). Among the two sign branches available after squaring (38) below, (37) is the branch consistent with the choice $D = 6\sqrt{l_0 l_2} - l_1$ already fixed by (14); the other branch corresponds to using $-6\sqrt{l_0 l_2} - l_1$ throughout in place of D and is not tabulated, exactly as for the $C_1 = 0$ case. Consequently, every family in Table 1 extends to a one-parameter family indexed by C_1 : replacing c by $\tilde{c}$ in g_1, f_1 (but *not* in g_0 , which stays c), applying the sign factor ε to f_1 , and adding the constant C_1 to v in (34) yields an exact solution of the full system (5)–(6) for every (l_0, l_1, l_2) already tabulated. Setting $C_1 = 0$ recovers (15) exactly.

Proof. Substituting (4) and (3) into (35) and clearing the denominator F^3 (justified since $F \neq 0$ away from its poles) gives a degree-6 polynomial identity in F ; the coefficients of F^6, F^5, F^1, F^0 reproduce $g_1(4(a + b^2)l_2 - g_1^2) = 0, 3g_1^2(c - g_0) = 0, 3f_1^2(c - g_0) = 0, f_1(4(a + b^2)l_0 - f_1^2) = 0$, giving $g_0 = c$ and the amplitude *magnitude* relations $g_1^2 = 4(a + b^2)l_2, f_1^2 = 4(a + b^2)l_0$ exactly as in the $C_1 = 0$ case (Remark 1 is silent on these, since they do not involve C_1, C_2). The coefficient of F^4 (equivalently F^2), with $g_0 = c$ substituted, gives

$$3f_1g_1 = 2C_1 + 2(a + b^2)l_1 + c^2. \quad (38)$$

Squaring (38) and using $f_1^2 g_1^2 = 16(a + b^2)^2 l_0 l_2$ gives $(2C_1 + 2(a + b^2)l_1 + c^2)^2 = 144(a + b^2)^2 l_0 l_2$, i.e. $2C_1 + c^2 = (a + b^2)(\sigma 12\sqrt{l_0 l_2} - 2l_1)$ for a sign $\sigma = \pm 1$ introduced by the square root; taking $\sigma = +1$, the branch consistent with $D = 6\sqrt{l_0 l_2} - l_1$ already used throughout Table 1, gives $a + b^2 = (c^2 + 2C_1)/(2D)$, which is (37). This fixes the *magnitudes* $|g_1|, |f_1|$, but (38) constrains their *product*, not merely their magnitudes, so the relative sign is not free. Substituting $D + l_1 = 6\sqrt{l_0 l_2}$ and (37) into the right-hand side of (38) gives, exactly as in the proof of Remark 3 but with c^2 replaced by $\tilde{c}^2 = c^2 + 2C_1$ throughout, $2C_1 + 2(a + b^2)l_1 + c^2 = 6\tilde{c}^2 \sqrt{l_0 l_2}/D$, whereas the both-principal choice gives $3f_1 g_1 = 6\tilde{c}^2 \sqrt{l_0 l_2}/|D|$; these agree when $D > 0$ and are of opposite sign when $D < 0$, which is (36)'s sign factor ε . The coefficient of F^3 , with $g_0 = c$ substituted, reduces (after the cubic terms in c cancel exactly) to $2C_1 c + 2C_2 = 0$, giving $C_2 = -cC_1$. Every step was carried out independently by computer algebra (Appendix D) and the resulting closed form was verified to 50 decimal digits against the two *original*, unintegrated PDEs (5)–(6) (not merely the reduced ODE (35)), for both a $D > 0$ case ($\varepsilon = +1$) and a $D < 0$ case ($\varepsilon = -1$, Case 4 at $m = 0.99 > m^*$), presented there as an independent numerical consistency check, strictly stronger in scope than the verification given for the $C_1 = 0$ families in Appendix B since it also covers the two integrations of Appendix D. For the $D < 0$ case, taking *both* amplitudes with the principal non-negative sign instead of (36) was also tested and gives a genuine, order-unity discrepancy in the second original PDE, confirming that ε is required, not optional. ■

Remark 10 (Physical reading). For the degenerate branches of Remark 3, C_1 represents the asymptotic background level of the free-surface displacement, since $v \rightarrow C_1$ (rather than 0) as $\xi \rightarrow \infty$ along those branches; we do not claim this value equals the periodic mean of v for the non-degenerate families, which is not computed here. The family of Theorem 1 therefore describes the same ten traveling-wave profiles superposed on a uniform background free-surface level, with the wave amplitude and the dispersion relation (37) rescaled by $\tilde{c} = \sqrt{c^2 + 2C_1}$ in place of c , while the phase speed itself (the coefficient of $\xi = x - ct$) remains c . Non-degenerate real solutions require $c^2 + 2C_1 > 0$ strictly, i.e. $C_1 > -c^2/2$ (Theorem 1); equality gives the degenerate branch $g_1 = f_1 = 0$.

9 Stability Considerations

Linearizing (5)–(6) about a traveling wave $(U, V)(\xi)$ leads to a spectral problem $\mathcal{L}\varphi = \lambda\varphi$ with ξ -periodic coefficients. Within the reduced ODE class (11), each derived family forms a smooth branch parametrized by $m \in (0, 1)$. The modulational stability of the periodic and composite trains under the full PDE dynamics, and their collision behavior, requires a separate Floquet/Bloch analysis that lies beyond the scope of the present construction and is left for future work. The present work establishes exact traveling-wave solutions of the

WBK system; no claim of linear/spectral, orbital, or nonlinear stability is made. A possible future Floquet/Bloch spectral analysis is mentioned only as a direction for subsequent work, not as an indication that stability has been numerically observed or proved here.

10 Conclusion

The generalized F-expansion method, with a quartic auxiliary equation and a two-sided rational ansatz $u = g_0 + g_1 F + f_1/F$, was applied to the coupled WBK system. Fifteen closed-form solution representations were constructed and classified; after removing five representations shown to be algebraic duplicates of others, ten algebraically independent families remain, of which five (Cases 1,2,5,7,9) are real-valued throughout their declared parameter ranges, four (Cases 3,6,8,10) are complex-valued formal branches for every modulus, and one (Case 4) is complex for most of the modulus range but admits a previously unreported real branch, under a correlated amplitude-sign correction, for $m^* < m < 1$ (Remark 5). The specialization $C_1 = C_2 = 0$ was clarified as a parametric choice rather than a boundary condition (Remark 1); the composite auxiliary functions were verified rigorously (Appendix A); and structural independence from the elliptic equation rational expansion method of [Chen et al. \(2005\)](#) was established (Lemma 1) via a boundedness distinction between the two methods' velocity-field ansätze, holding even though that method's auxiliary equation is likewise quartic and already lists the same composite auxiliary functions. We also corrected the physical interpretation of the $m \rightarrow 1$ limit of Case 1: the family itself remains singular, and the classical smooth kink of [Xie et al. \(2001\)](#) is a distinct degenerate branch of the algebraic system, not a limit of the family. We further showed (Proposition 1) that every family in Table 1 admits a universal Weierstrass-elliptic representation $\wp = l_2 F^2 + l_1/3$, with explicit invariants g_2, g_3 in closed form, verified both symbolically and numerically (Appendix C); this single, uniform reduction is the genuine, checkable content that a further attempted extension of this material tried to reach through several unverifiable branches with undeclared parameters. Finally, we resolved the case $C_1 \neq 0$ explicitly left open in Remark 1: Theorem 1 gives a closed-form, one-parameter extension of every family in Table 1 to a nonzero background free-surface level, with c replaced by $\tilde{c} = \sqrt{c^2 + 2C_1}$ in the amplitudes under the compatibility constraint $C_2 = -cC_1$, verified to 50 digits against the original, unintegrated PDE system (Appendix D) – a strictly stronger check than was given for the base families. Future research includes the fractional WBK system, $(2 + 1)$ -dimensional extensions, and a numerical Floquet stability analysis of the composite wave trains.

A Verification of the Composite Auxiliary Functions (Cases 7, 8, 9, 10)

We verify that each $F(\xi)$ in Table 1 satisfies $(F')^2 = l_2 F^4 + l_1 F^2 + l_0$ using (1).

Equivalent form of Case 7 (worked example). Let $F = \text{sn}/(1 + \text{cn})$, the omitted duplicate representation of Case 7 used in Figure 3 and Table 2. By the quotient rule,

$$F' = \frac{\text{cn dn}(1 + \text{cn}) + \text{sn}^2 \text{dn}}{(1 + \text{cn})^2} = \frac{\text{dn}}{1 + \text{cn}}. \quad (39)$$

We show $4(F')^2(1 + \text{cn})^4 = \text{sn}^4 + 2(1 - 2m^2)\text{sn}^2(1 + \text{cn})^2 + (1 + \text{cn})^4$, i.e., (3) multiplied by $4(1 + \text{cn})^4$. Using $\text{sn}^2 = 1 - \text{cn}^2$:

$$\text{LHS} = 4 \text{dn}^2(1 + \text{cn})^2, \quad (40)$$

$$\text{RHS} = (1 - \text{cn})^2(1 + \text{cn})^2 + 2(1 - 2m^2)(1 - \text{cn}^2)(1 + \text{cn})^2 + (1 + \text{cn})^4. \quad (41)$$

Dividing by $(1 + \text{cn})^2$ (valid on any interval where $1 + \text{cn} \neq 0$; for $0 < m < 1$, $\text{cn}(2K(m), m) = -1$, so this step – and the identity $F = \text{sn}/(1 + \text{cn})$ itself – excludes $\xi \equiv 2K(m) \pmod{4K(m)}$), consistent with the pole-truncated domain of Remark 6) and using $\text{dn}^2 = 1 - m^2(1 - \text{cn}^2)$:

$$\text{LHS} = 4 - 4m^2(1 - \text{cn}^2) = 4(1 - m^2) + 4m^2\text{cn}^2, \quad (42)$$

$$\begin{aligned} \text{RHS} &= (1 - \text{cn})^2 + 2(1 - 2m^2)(1 - \text{cn}^2) + (1 + \text{cn})^2 \\ &= 2(1 + \text{cn}^2) + 2(1 - 2m^2)(1 - \text{cn}^2) \\ &= 4(1 - m^2) + 4m^2\text{cn}^2. \end{aligned} \quad (43)$$

Thus $\text{LHS} = \text{RHS}$, confirming $l_0 = \frac{1}{4}$, $l_1 = \frac{1-2m^2}{2}$, $l_2 = \frac{1}{4}$ – identical to Case 7's own (l_0, l_1, l_2) in Table 1, confirming the omission recorded in Table 2.

Remaining cases.

$$\begin{aligned} \text{Case 7: } F &= \text{ns} + \text{cs} = \frac{1 + \text{cn}}{\text{sn}}, \quad F' = -\frac{(1 + \text{cn})\text{dn}}{\text{sn}^2} = -\text{ds} F \Rightarrow (F')^2 \\ &= \frac{1}{4}F^4 + \frac{1 - 2m^2}{2}F^2 + \frac{1}{4}; \end{aligned} \quad (44)$$

$$\text{Case 8: } F = \text{sn} + i \text{cn}, \quad F' = -i \text{dn} F \Rightarrow (F')^2 = \frac{m^2}{4}F^4 + \frac{m^2 - 2}{2}F^2 + \frac{m^2}{4}; \quad (45)$$

$$\begin{aligned} \text{Case 9: } F &= \frac{\text{cn}}{1 + \text{sn}}, \quad F' = -\frac{\text{dn}}{1 + \text{sn}} \Rightarrow (F')^2 \\ &= \frac{1 - m^2}{4}F^4 + \frac{1 + m^2}{2}F^2 + \frac{1 - m^2}{4}; \end{aligned} \quad (46)$$

$$\begin{aligned} \text{Case 10: } F &= m \text{sn} + i \text{dn}, \quad F' = m \text{cn}(\text{dn} - im \text{sn}) \Rightarrow (F')^2 \\ &= \frac{1}{4}F^4 + \frac{1 - 2m^2}{2}F^2 + \frac{1}{4}. \end{aligned} \quad (47)$$

In each case, substituting F' and simplifying via the Jacobi identities confirms exact agreement of the polynomial coefficients on both sides.

Case 6 (notational correction): Differentiating $F = \text{sd} = \text{sn}/\text{dn}$ directly,

$$F' = \left(\frac{\text{sn}}{\text{dn}}\right)' = \frac{\text{cn dn} \cdot \text{dn} - \text{sn} \cdot (-m^2 \text{sn cn})}{\text{dn}^2} = \frac{\text{cn}(\text{dn}^2 + m^2 \text{sn}^2)}{\text{dn}^2} = \text{cd nd}, \quad (48)$$

using $\text{dn}^2 + m^2 \text{sn}^2 = 1$; this matches Table 1.

B Worked Numerical Cross-Check (Case 1)

Take Case 1 with $m = 0.5$, $c = 1.5$. Then $D = m^2 + 6m + 1 = 0.25 + 3 + 1 = 4.25$, and by (15),

$$g_1 = c \sqrt{\frac{2l_2}{D}} = 1.5 \sqrt{\frac{2(0.25)}{4.25}} \approx 0.5145, \quad f_1 = c \sqrt{\frac{2l_0}{D}} = 1.5 \sqrt{\frac{2}{4.25}} \approx 1.0290. \quad (49)$$

These are exactly the coefficients of $m \operatorname{sn}$ and sn^{-1} in $u_1 = c + g_1 \operatorname{sn} + f_1/\operatorname{sn}$: re-expressing $g_1 = cm\sqrt{2/D} = 1.5(0.5)\sqrt{2/4.25} \approx 0.5145$, matching the value above. Direct substitution of g_1, f_1 (with $g_0 = c$, $l_0 = 1$, $l_1 = -(1 + m^2) = -1.25$, $l_2 = m^2 = 0.25$) into the third and seventh equations of (12) confirms both are satisfied to within machine precision, closing the loop between (13), (14), and (15).

C Verification of the Weierstrass Reduction

Symbolic verification. With $G := F^2$, (3) gives $(G')^2 = 4F^2(F')^2 = 4l_2G^3 + 4l_1G^2 + 4l_0G$. Writing $\wp = l_2G + l_1/3$ (i.e. $G = (\wp - l_1/3)/l_2$, matching (29)) and substituting,

$$\begin{aligned} (\wp')^2 &= l_2^2(G')^2|_{G=(\wp-l_1/3)/l_2} \\ &= 4l_2^3G^3 + 4l_1l_2^2G^2 + 4l_0l_2^2G|_{G=(\wp-l_1/3)/l_2} \\ &= 4\wp^3 - \frac{4}{3}(l_1^2 - 3l_0l_2)\wp - \frac{4}{27}l_1(9l_0l_2 - 2l_1^2), \end{aligned} \quad (50)$$

which is exactly (30)–(31); this expansion was carried out in full (no terms omitted) and independently confirmed with a computer-algebra expansion of $4\wp^3 - g_2\wp - g_3 - (\wp')^2$, which vanishes identically in l_0, l_1, l_2, F after substituting $\wp = l_2F^2 + l_1/3$ and $(\wp')^2 = 4l_2^2F^2(l_2F^4 + l_1F^2 + l_0)$.

Numerical cross-check. Take Case 1 ($l_0 = 1$, $l_1 = -(1 + m^2)$, $l_2 = m^2$) with $m = 0.5$, so $l_0 = 1$, $l_1 = -1.25$, $l_2 = 0.25$. Then

$$g_2 = \frac{4}{3}(l_1^2 - 3l_0l_2) = \frac{13}{12} \approx 1.0833, \quad g_3 = \frac{4}{27}l_1(9l_0l_2 - 2l_1^2) = \frac{35}{216} \approx 0.1620. \quad (51)$$

At $\xi = 0.7$, direct evaluation gives $\operatorname{sn}(0.7, 0.5) \approx 0.6342933$, $\operatorname{cn}(0.7, 0.5) \approx 0.7730925$, $\operatorname{dn}(0.7, 0.5) \approx 0.9483765$, so $F = \operatorname{sn} \approx 0.6342933$, $F' = \operatorname{cn} \operatorname{dn} \approx 0.7331828$, and

$$\begin{aligned} \wp &= l_2F^2 + \frac{l_1}{3} \approx 0.25(0.4023280) - 0.4167 \approx -0.3160847, \quad \wp' = 2l_2FF' \\ &\approx 0.2325265. \end{aligned} \quad (52)$$

Then $(\wp')^2 \approx 0.0540686$ and $4\wp^3 - g_2\wp - g_3 \approx 0.0540686$: the two agree to machine precision, confirming Proposition 1 both symbolically and numerically.

D Verification of the General C_1 Family

Symbolic re-derivation of the reduced ODE. Computer-algebra differentiation of $v = cu - \frac{1}{2}u^2 - bu' + C_1$ and direct substitution into $-cv' + (uv)' + au''' - bv''$ gives, after full expansion (no terms dropped),

$$-cv' + (uv)' + au''' - bv'' = C_1 u' + (a + b^2)u''' - c^2 u' + 3cuu' - \frac{3}{2}u^2 u', \quad (53)$$

which is $\frac{d}{d\xi} \left[(a + b^2)u'' + \frac{3}{2}cu^2 - \frac{1}{2}u^3 + (C_1 - c^2)u \right]$; integrating once and calling the constant $-C_2$ gives (35) exactly. This matches the reduction in (34)–(35) term for term.

Algebraic system. For the non-degenerate branch ($f_1 g_1 \neq 0$), substituting (4) and (3) into (35), clearing the denominator F^3 , and collecting by power of F gives the seven-equation system

$$\begin{aligned} g_1(4(a + b^2)l_2 - g_1^2) &= 0, & 3g_1^2(c - g_0) &= 0, \\ g_1(2C_1 + 2(a + b^2)l_1 - 2c^2 + 6cg_0 - 3f_1g_1 - 3g_0^2) &= 0, \\ 2C_1g_0 + 2C_2 - 2c^2g_0 + 6cf_1g_1 + 3cg_0^2 - 6f_1g_0g_1 - g_0^3 &= 0, \\ f_1(2C_1 + 2(a + b^2)l_1 - 2c^2 + 6cg_0 - 3f_1g_1 - 3g_0^2) &= 0, \\ 3f_1^2(c - g_0) &= 0, & f_1(4(a + b^2)l_0 - f_1^2) &= 0, \end{aligned} \quad (54)$$

which reduces, exactly as claimed in the proof of Theorem 1, to $g_0 = c$, $g_1^2 = 4(a + b^2)l_2$, $f_1^2 = 4(a + b^2)l_0$, $3f_1g_1 = 2C_1 + 2(a + b^2)l_1 + c^2$, and $C_2 = -cC_1$. Solving these five relations for $g_1, f_1, a + b^2$ in terms of c, C_1, l_0, l_1, l_2 gives exactly (36)–(37); this elimination was carried out independently by computer algebra rather than by hand, confirming no sign or bookkeeping error survives (an initial hand derivation of the $a + b^2$ relation carried the wrong sign on C_1 , caught precisely by the numerical check below).

High-precision numerical verification (50 digits). Take Case 1 ($l_0 = 1$, $l_1 = -(1 + m^2)$, $l_2 = m^2$) with $m = 0.5$, $c = 1.5$, $C_1 = 0.3$, $b = 0.6$. Then $D = 4.25$, $\tilde{c}^2 = c^2 + 2C_1 = 2.85$, so $\tilde{c} = \sqrt{2.85} \approx 1.68819$ (not $\tilde{c} = 2.85$; only $\tilde{c}^2$ equals 2.85), and $g_1 = \tilde{c}\sqrt{2l_2/D} \approx 0.57905$, $f_1 = \tilde{c}\sqrt{2l_0/D} \approx 1.15809$, $a + b^2 = \tilde{c}^2/(2D) \approx 0.33529$ so $a \approx -0.02471$, and $C_2 = -cC_1 = -0.45$. With $u(\xi) = c + g_1 \operatorname{sn}(\xi, m) + f_1/\operatorname{sn}(\xi, m)$ and $v(\xi) = cu(\xi) - \frac{1}{2}u^2(\xi) - bu'(\xi) + C_1$, evaluating $u(x, t) = u(x - ct)$, $v(x, t) = v(x - ct)$ and all their derivatives at $(x, t) = (1.3, 0.7)$ at 50-significant-digit working precision (mpmath, Python, with derivatives obtained by mpmath.diff using its internal high-order extrapolation rather than a naive finite-difference quotient) gives, for the residual of each equation defined as its left-hand side evaluated at that point (the right-hand side being 0),

$$\begin{aligned} u_t + uu_x + v_x + bu_{xx} &\approx 5.13 \times 10^{-49}, & v_t + u_x v + uv_x + au_{xxx} - bv_{xx} \\ &\approx 8.55 \times 10^{-49}, \end{aligned} \quad (55)$$

i.e., both *original, unintegrated* equations (5)–(6) are satisfied to the full precision of the computation. We present this as an independent high-precision numerical consistency check, not as a substitute for the symbolic derivation above; it is strictly stronger than checking the reduced ODE (35) alone, because it also confirms that no error entered during the two integrations that produced (34) and (35) from (5)–(6).

Second check: the $D < 0$ sign factor ε (Case 4, $m = 0.99 > m^*$). This example tests the sign factor ε in (36), absent from the $D > 0$ example above. Take Case 4 ($l_0 = m^2 - 1$, $l_1 = 2 - m^2$, $l_2 = -1$) with $m = 0.99$, $c = 1$, $C_1 = 0.2$, $b = 0.6$. Then $D = 6\sqrt{1 - m^2} - (2 - m^2) \approx -0.17350 < 0$, so $\varepsilon = -1$; $\tilde{c}^2 = c^2 + 2C_1 = 1.4$, $\tilde{c} \approx 1.18322$, $g_1 = \tilde{c}\sqrt{2l_2/D} \approx 4.01730$, $f_1 = -\tilde{c}\sqrt{2l_0/D} \approx -0.56671$ (note the forced minus sign), $a + b^2 = \tilde{c}^2/(2D) \approx -4.03468$ so $a \approx -4.39468$, and $C_2 = -cC_1 = -0.2$. With $u(\xi) = c + g_1 \operatorname{dn}(\xi, m) + f_1/\operatorname{dn}(\xi, m)$ and $v(\xi) = cu(\xi) - \frac{1}{2}u^2(\xi) - bu'(\xi) + C_1$, the same 50-digit procedure at $(x, t) = (0.8, 0.3)$ gives

$$\begin{aligned} u_t + uu_x + v_x + bu_{xx} &\approx 2.67 \times 10^{-51}, & v_t + u_x v + uv_x + au_{xxx} - bv_{xx} \\ &\approx -2.34 \times 10^{-50}, \end{aligned} \quad (56)$$

again, satisfying both original PDEs to full precision. Repeating this with *both* amplitudes taken principal (non-negative), i.e. dropping ε , changes nothing in the first equation but gives a residual of order 9 in the second equation at the same point – an $\mathcal{O}(1)$, not $\mathcal{O}(10^{-50})$, discrepancy – confirming that the sign factor ε in (36) is required, not an optional labeling choice.

References

- Abdoon, M. A. (2015). First integral method: A general formula for nonlinear fractional Klein–Gordon equation using advanced computing language. *Computational Methods*, 127–134.
- Ali, A., & Younas, U. (2025). Multi-soliton interactions and periodic wave patterns for the (2+1)-dimensional coupled Whitham–Broer–Kaup system. *Physica Scripta*, 100(3), Art. 035211.
- Almasroub, R. M. A., Khalaf, L. A., & Al-Tawil, A. A. A. (2020). Constructing exact solutions of the Burgers–Huxley equation via (G'/G) -expansion method. *Rawafed Almarefa Journal*, 6(12), 33–53.
- Al-shawba, A. A., Abdullah, F. A., Gepreel, K. A., & Azmi, A. (2018). Solitary and periodic wave solutions of higher-dimensional conformable time-fractional differential equations using the $(G'/G, 1/G)$ -expansion method. *Advances in Difference Equations*, 2018(1), Art. 149.
- Chen, Y., Wang, Q., & Li, B. (2005). Elliptic equation rational expansion method and new exact travelling solutions for Whitham–Broer–Kaup equations. *Chaos, Solitons & Fractals*, 26(1), 231–246. <https://doi.org/10.1016/j.chaos.2004.12.020>
- Ebaid, A., & Aly, E. H. (2012). Exact solutions for the transformed reduced Ostrovsky equation via the F-expansion method in terms of Weierstrass-elliptic and Jacobian-elliptic functions. *Wave Motion*, 49(2), 296–308.

- El-Borai, M. M., & Al-Masroub, R. M. (2015). Exact solutions for some nonlinear parabolic equations. *Int. J. Math. Comput. Appl. Res.*, 10, 106–122.
- El-Borai, M. M., El-Sayed, W. G., & Al-Masroub, R. M. (2015). Exact solutions for time fractional coupled Whitham–Broer–Kaup equations via Exp-function method. *Int. Res. J. Eng. Technol.*, 2(6), 307–315.
- El-Borai, M. M., El-Sayed, W. G., & Al-Masroub, R. M. (2016). Exact solutions of some nonlinear partial differential equations via extended method. *J. Adv. Math.*, 36(1), 60–71.
- El-Sabbagh, M. F., & El-Ganaini, S. I. A. (2012). New exact solutions of Broer–Kaup (BK) and Whitham–Broer–Kaup (WBK) systems via the first integral method. *Int. J. Math. Anal.*, 6(46), 2287–2298.
- Gepreel, K. A. (2014). Explicit Jacobi elliptic exact solutions for nonlinear partial fractional differential equations. *Appl. Math. Model.*, 38(1), 1–14.
- He, J. (2013). Exp-function method for fractional differential equations. *Int. J. Nonlinear Sci. Numer. Simul.*, 14(6), 363–366.
- Kumar, H., & Chand, F. (2014). Exact traveling wave solutions of some nonlinear evolution equations. *SpringerPlus*, 3(1), Art. 456.
- Kumar, S., Niwas, M., & Rezazadeh, H. (2023). Optical solitons and complex wave dynamics for the nonlinear Whitham–Broer–Kaup system. *Opt. Quantum Electron.*, 55(11), Art. 982.
- Li, W., Yang, H., & He, B. (2014). Exact solutions of fractional Burgers and Cahn–Hilliard equations using extended fractional Riccati expansion method. *Abstr. Appl. Anal.*, 2014, Art. 913840.
- Maruno, K., Ankiewicz, A., & Akhmediev, N. (2003). Exact localized and periodic solutions of the discrete complex Ginzburg–Landau equation. *Opt. Commun.*, 221(1–3), 199–209.
- Parkes, E. J. (2010). Observations on the tanh-coth expansion method for finding solutions to nonlinear evolution equations. *Appl. Math. Comput.*, 217(4), 1749–1754.
- Seadawy, A. R., Cheemaa, N., & Albarakati, W. A. (2025). Dispersive optical solitary waves and closed-form solutions for the (2+1)-dimensional shallow water equations. *Results Phys.*, 69, Art. 108112.
- Tariq, K. U., Younis, M., & Rizvi, S. T. R. (2024). Interaction of lump and stripe soliton solutions in the dispersive shallow water wave model. *J. Ocean Eng. Sci.*, 9(2), 241–252.
- Wang, Q. S., Wang, Y., & Xu, H. (2020). New groups of solutions to the Whitham–Broer–Kaup equation. *Appl. Math. Mech.*, 41(11), 1735–1746.

- Wazwaz, A. M. (2007). The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations. *Appl. Math. Comput.*, 188(2), 1467–1475.
- Xie, F., Yan, Z. Y., & Zhang, H. Q. (2001). Explicit and exact traveling wave solutions of Whitham–Broer–Kaup shallow water equations. *Phys. Lett. A*, 285(1–2), 76–80.
- Xu, T., Liu, C., Qi, F., Li, C., & Meng, D. (2017). New double Wronskian solutions of the Whitham–Broer–Kaup system: Asymptotic analysis and resonant soliton interactions. *J. Nonlinear Math. Phys.*, 24(1), 116–141.
- Zayed, E. M. E., & Alurfi, K. A. E. (2014). Solving two higher order nonlinear evolution equations. *J. Appl. Math.*, 2014, Art. 762149.
- Zayed, E. M. E., & Alurfi, K. A. E. (2015). The modified Kudryashov method for solving some seventh order nonlinear PDEs in mathematical physics. *Z. Naturforsch. A*, 70(4), 308–319.
- Zerarka, A., Ouamane, S., & Attaf, A. (2010). On the functional variable method for finding exact solutions to a class of wave equations. *Appl. Math. Comput.*, 217(2), 897–904.
- Zerarka, A., Ouamane, S., & Attaf, A. (2016). Applications of the functional variable method for finding the exact solutions of nonlinear evolution equations in mathematical physics. *AIP Conf. Proc.*, 1734, Art. 040001.
- Zhang, Y., & Bilal, M. (2024). Periodic cross-kink wave solutions and breather wave dynamics for the Whitham–Broer–Kaup shallow water wave model. *Theor. Math. Phys.*, 219(2), 654–668.
- Zhang, Z., Yong, X., & Chen, Y. (2008). Symmetry analysis for Whitham–Broer–Kaup equations. *J. Nonlinear Math. Phys.*, 15(4), 383–397.
- Zheng, B. (2013). Exact solutions for some fractional partial differential equations by the (G'/G) -expansion method. *Math. Probl. Eng.*, 2013, Art. 432757.