Generalizations of Hadamard Products of Certain Meromorphic Uniformly Univalent Functions with Positive Coefficients

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Abstract

In this paper, we obtained some results concerning the generalized Hadamard products of certian meromorphic uniformly starlike and convex functions with positive coefficients.

Key words and phrases: Univalent, meromorphic, starlike, convex, uniformly, Hadamard product.

1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k, \tag{1.1}$$

which are analytic and univalent in the punctured unit disc

$U^*=\{z : z\in \mathbb{C} \text{ and } 0<|z|<1\}$.

A function $f(z)\in\Sigma$ is meromorhpically starlike of order α if

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\left(z\in U^*;0\le\alpha<1\right). \tag{1.2}$$

We denote by $\Sigma S^*(\alpha)$ the class of all meromorphically starlike functions of order α.

A function $f(z)$ in Σ is said to be meromorphically convex of order α if

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\left(z\in U^*;0\le\alpha<1\right). \tag{1.3}$$

And we denote by $\Sigma K(\alpha)$ the class of all meromorphically convex functions of order α.

The classes $\Sigma S^*(\alpha)$ and $\Sigma K(\alpha)$ have been studied by Aouf and Silverman [3], Pomerenke [12], Clunie [6], Kaczmarski [9], Royster [13], Juneja and Reddy [8], Mogra [11] and others.

A function f of the form (1.1) is said to be in the class $U\Sigma S^*(\alpha,\beta)$ of meromorphic uniformly β-starlike functions of order α if it satisfies the condition:

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}+\alpha\right\}>\beta\left|\frac{zf'(z)}{f(z)}+1\right|\left(z\in U;0\le\alpha<1;\beta\ge 0\right). \tag{1.4}$$

Also a function f of the form (1.1) is said to be in the class $U\Sigma K(\alpha,\beta)$ of meromorphic

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uniformly β-convex functions of order α if it satisfies the condition:

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}+\alpha\right\}>\beta\left|2+\frac{zf''(z)}{f'(z)}\right|\left(z\in U;0\le\alpha<1;\beta\ge 0\right). \tag{1.5}$$

It follows from (1.4) and (1.5) that

$$f\in U\Sigma K(\alpha,\beta)\Leftrightarrow -zf'\in U\Sigma S^*(\alpha,\beta). \tag{1.6}$$

The classes $U\Sigma S^*(\alpha,\beta)$ and $U\Sigma K(\alpha,\beta)$ have been studied by Aouf et al. [1], Atshan and Kulkarni [4], and others.

We note that:

(i) $U\Sigma S^*(\alpha,0)=S_n^*(\alpha)$ and $U\Sigma K(\alpha,0)=K_n(\alpha)$ [see Aouf and Silverman [3], with $n=1$];

(ii) $U\Sigma S^*(\alpha,0)=\Sigma_p S_n^*(\alpha,\gamma)$ and $U\Sigma K(\alpha,0)=\Sigma_p K_n(\alpha,\gamma)$ [also see El-Ashwah et al. [7], with $n=p=\gamma=1$].

From [1, Theorem 1, with $\lambda=0$ and $g(z)=\frac{1}{z(1-z)}$] we can obtain the following coefficients inquality, for the class $U\Sigma S^*(\alpha,\beta)$.

Lemma 1 [1]. Let the function f defined by (1.1). Then $f\in U\Sigma S^*)$ if and only if

$$\sum_{k=1}^{\infty}[k(1+\beta)+(\alpha+\beta)]a_k\le(1-\alpha). \tag{1.7}$$

From (1.6) and (1.7) we can obtain the following lemma.

Lemma 2. Let the function f defined by (1.1). Then $f\in U\Sigma K\left(\alpha,\beta\right)$ if and only if

$$\sum_{k=1}^{\infty}k[k(1+\beta)+\left(\alpha+\beta\right)]a_k\le(1-\alpha). \tag{1.8}$$

Remark 1.

(i) Putting $\beta=0$ in Lemma 1, we obtain the result obtained by Aouf and Silverman [3, Lemma 1, with $n=1$];

(ii) Putting $\beta=0$ in Lemma 1, we obtain the result obtained by El-Ashwah et al. [7, Lemma 1, with $n=p=\gamma=1$];

(iii) Putting $\beta=0$ in Lemma 2, we obtain the result obtained by Aouf and Silverman [3, Lemma 2, with $n=1$];

(vi) Putting $\beta=0$ in Lemma 2, we obtain the result obtained by El-Ashwah et al. [7, Lemma 2, with $n=p=\gamma=1$].

For the functions

$$f_j(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_{k,j}z^k\left(a_{k,j}\ge 0;j=1,2\right). \tag{1.9}$$

We denote by $\left(f_1*f_2\right)(z)$ the Hadamard product (or convolution) of functions $f_1(z)$ and

$f_2(z)$, that is

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k. \qquad (1.10)$$

For any real number r and s, we define the generalized Hadamard product $(f_1 \Delta f_2)(r,s;z)$ by [see Aouf and Silverman [3]]

$$(f_1 \Delta f_2)(r,s;z) = \frac{1}{z} + \sum_{k=1}^{\infty} (a_{k,1})^r (a_{k,2})^s z^k. \qquad (1.11)$$

Note that, if we take $r = s = 1$, then we have
$(f_1 \Delta f_2)(1,1;z) = (f_1 * f_2)(z) \, (z \in U^*)$.

Further for functions $f_j(z)(j=1,2)$ are given by (1.9), the familiar Hölder inequality assumes the following form (see [10,14]).

$$\sum_{k=1}^{\infty}\left(\prod_{j=1}^{m} a_{k,j}\right) \le \prod_{j=1}^{m}\left(\sum_{k=1}^{\infty}(a_{k,j})^{s_j}\right)^{\frac{1}{s_j}} \quad (s_j \ge 1;\, j=1,2,\dots,m\,;\, \sum_{j=1}^{m}\frac{1}{s_j} \ge 1). \qquad (1.12)$$

2. Main Results

Theorem 1. If the functions $f_j(j=1,2)$ defined by (1.9) are in the classes $U\Sigma S^*(\alpha_j,\beta)$, for each j, then

$(f_1 \Delta f_2)\left(\frac{1}{r}, \frac{r-1}{r}; z\right) \in U\Sigma S^*(\xi,\beta)(r>1)$,

and

$$\xi \le 1 - \frac{(k+1)(1+\beta)}{1 + \left[\frac{k(1+\beta)+(\alpha_1+\beta)}{1-\alpha_1}\right]^{\frac{1}{r}} \left[\frac{k(1+\beta)+(\alpha_2+\beta)}{1-\alpha_2}\right]^{\frac{r-1}{r}}}. \qquad (2.1)$$

Proof. Since $f_j(z) \in U\Sigma S^*(\alpha_j,\beta)(j=1,2)$, by using Lemma 1, we have

$$\sum_{k=1}^{\infty} \frac{[k(1+\beta)+(\alpha_j+\beta)]a_{k,j}}{(1-\alpha_j)} \le 1 \, (j=1,2). \qquad (2.2)$$

Moreover,

$$\left\{\sum_{k=1}^{\infty} \frac{[k(1+\beta)+(\alpha_1+\beta)]a_{k,1}}{(1-\alpha_1)}\right\}^{\frac{1}{r}} \le 1, \qquad (2.3)$$

$$\left\{\sum_{k=1}^{\infty} \frac{[k(1+\beta)+(\alpha_2+\beta)]a_{k,2}}{(1-\alpha_2)}\right\}^{\frac{r-1}{r}} \le 1. \qquad (2.4)$$

By using Holder inequality, we get

$$\sum_{k=1}^{\infty}\left\{\left[\frac{[k(1+\beta)+(\alpha_1+\beta)]}{(1-\alpha_1)}\right]^{\frac{1}{r}}\left[\frac{[k(1+\beta)+(\alpha_2+\beta)]}{(1-\alpha_2)}\right]^{\frac{r-1}{r}}\right\}\left(a_{k,1}\right)^{\frac{1}{r}}\left(a_{k,2}\right)^{\frac{r-1}{r}}\leq 1. \qquad (2.5)$$

Since

$$\left(f_1\Delta f_2\right)\left(\frac{1}{r},\frac{r-1}{r};z\right)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(a_{k,1}\right)^{\frac{1}{r}}\left(a_{k,2}\right)^{\frac{r-1}{r}}z^k. \qquad (2.6)$$

Therefore, we need to find the largest ξ, such that,

$$\sum_{k=1}^{\infty}\frac{[k(1+\beta)+(\xi+\beta)]}{(1-\xi)}\left(a_{k,1}\right)^{\frac{1}{r}}\left(a_{k,2}\right)^{\frac{r-1}{r}}\leq 1, \qquad (2.7)$$

that is

$$\frac{[k(1+\beta)+(\xi+\beta)]}{(1-\xi)}\leq\left[\frac{[k(1+\beta)+(\alpha_1+\beta)]}{(1-\alpha_1)}\right]^{\frac{1}{r}}\left[\frac{[k(1+\beta)+(\alpha_2+\beta)]}{(1-\alpha_2)}\right]^{\frac{r-1}{r}}, \qquad (2.8)$$

which implies

$$\xi\leq 1-\frac{(k+1)(1+\beta)}{1+\left[\frac{k(1+\beta)+(\alpha_1+\beta)}{1-\alpha_1}\right]^{\frac{1}{r}}\left[\frac{k(1+\beta)+(\alpha_2+\beta)}{1-\alpha_2}\right]^{\frac{r-1}{r}}}. \qquad (2.9)$$

This completes the proof of Theorem 1.

Remark 2.

(i) Putting $\beta=0$ in Theorem 1, we obtain the result obtained by Aouf and Silverman [3, Theorem 1, with $n=1$];

(ii) Putting $\beta=0$ in Theorem 1, we obtain the result obtained by El-Ashwah et al. [7, Theorem 1, with $n=p=\gamma=1$] .

Corollary 1. If the functions $f_j(z)(j=1,2)$ defined by (1.9) are in the class $U\Sigma S^*(\alpha,\beta)$, then $\left(f_1\Delta f_2\right)\left(\frac{1}{r},\frac{r-1}{r};z\right)\in U\Sigma S^*(\alpha,\beta)(r>1)$.

Proof. In view of Lemma 1, Corollary 1 follows immediately from Theorem 1 by taking $\alpha_j=\alpha(j=1,2)$.

Theorem 2. Let the functions $f_j(z)(j=1,2)$ defined by (1.9) are in the classes $U\Sigma K(\alpha_j,\beta)$ for each j, then $\left(f_1\Delta f_2\right)\left(\frac{1}{r},\frac{r-1}{r};z\right)\in U\Sigma K(\xi,\beta)$,

where $r>1$ and ξ is defined by (2.1).

Proof. Since $f_j(z)\in U\Sigma K(\alpha_j,\beta)(j=1,2)$, by using Lemma 2, we get

$$\sum_{k=1}^{\infty}\frac{k\,[k\,(1+\beta)+\left(\alpha_j+\beta\right)]}{(1-\alpha_j)}a_{k,j}\le 1 \quad \left(j=1,2\right). \tag{2.10}$$

Thus, the proof of Theorem 2 is similar to that of Theorem 1, where Lemma 2 is used instead of Lemma 1.

Remark 3.

(i) Putting $\beta=0$ in Theorem 2, we obtain the result obtained by Aouf and Silverman [3, Theorem 2, with $n=1$];

(ii) Putting $\beta=0$ in Theorem 2, we obtain the result obtained by El-Ashwah et al. [7, Theorem 2, with $n=p=\gamma=1$] .

Corollary 2. Let the functions $f_j\left(z\right)\left(j=1,2\right)$ defined by (1.9) are in the class $U\Sigma K\left(\alpha,\beta\right)$, then

$$\left(f_1\Delta f_2\right)\left(\frac{1}{r},\frac{r-1}{r};z\right)\in U\Sigma K\left(\alpha,\beta\right) \quad \left(r>1\right).$$

Theorem 3. Let the functions $f_j\left(z\right)\left(j=1,2,...m\right)$ defined by (1.9) be in the classes $U\Sigma S^*(\alpha_j,\beta)$ for each j, and let the function $F_m\left(z\right)$ defined by

$$F_m\left(z\right)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m}(a_{k,j})^r\right)z^k \qquad \left(z\in U^*;r\ge 2\right). \tag{2.11}$$

Then $F_m\left(z\right)\in U\Sigma S^*(\xi_m,\beta)$, where

$$\xi_m\le 1-\frac{2m\left(\beta+1\right)\left(1-\alpha\right)^r}{m\left(1-\alpha\right)^r+(1+\alpha+2\beta)^r}, \tag{2.12}$$

$$\alpha=\min_{1\le j\le m}\left\{\alpha_j\right\}, \tag{2.13}$$

and

$$\left(1+\alpha+2\beta\right)^r>m\left(1+2\beta\right)\left(1-\alpha\right)^r. \tag{2.14}$$

Proof. Since $f_j\left(z\right)\in U\Sigma S^*(\alpha_j,\beta)$, using Lemma 1, we obtain

$$\sum_{k=1}^{\infty}\frac{[k\,(1+\beta)+(\alpha_j+\beta)]a_{k,j}}{\left(1-\alpha_j\right)}\le 1 \qquad (j=1,2,...,m) \tag{2.15}$$

and

$$\sum_{k=1}^{\infty}\left\{\frac{[k\,(1+\beta)+(\alpha_j+\beta)]}{\left(1-\alpha_j\right)}\right\}^r(a_{k,j})^r\le\left\{\sum_{k=1}^{\infty}\frac{[k\,(1+\beta)+(\alpha_j+\beta)]a_{k,j}}{\left(1-\alpha_j\right)}\right\}^r\le 1. \tag{2.16}$$

It follows from (2.16) that

$$\sum_{k=1}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left[\frac{[k(1+\beta)+(\alpha_j+\beta)]}{(1-\alpha_j)}\right]^r (a_{k,j})^r\right\}\le 1. \tag{2.17}$$

By virtue of (2.17), we find that

$$\sum_{k=1}^{\infty}\left(\frac{[k(1+\beta)+(\xi_m+\beta)]}{(1-\xi_m)}\right)\left(\sum_{j=1}^{m}(a_{k,j})^r\right)\le \sum_{k=1}^{\infty}\frac{1}{m}\left[\frac{[k(1+\beta)+(\alpha+\beta)]}{(1-\alpha)}\right]^r\left(\sum_{j=1}^{m}(a_{k,j})^r\right)$$

$$\le \sum_{k=1}^{\infty}\frac{1}{m}\sum_{j=1}^{m}\left[\frac{[k(1+\beta)+(\alpha_j+\beta)]}{(1-\alpha_j)}\right]^r (a_{k,j})^r\le 1, \tag{2.18}$$

which implies that

$$\xi_m\le 1-\frac{m(k+1)(\beta+1)(1-\alpha)^r}{m(1-\alpha)^r+[k(1+\beta)+(\alpha+\beta)]^r},\quad (k\ge 1). \tag{2.19}$$

Now let

$$g(k)=1-\frac{m(k+1)(\beta+1)(1-\alpha)^r}{m(1-\alpha)^r+[k(1+\beta)+(\alpha+\beta)]^r}\ . \tag{2.20}$$

Then

$$g'(k)=\left\{m(1-\alpha)^r+[k(1+\beta)+(k+\beta)]^r\right\}\left(-m(1-\alpha)^r(\beta+1)\right).$$

$$.\ \left\{m(1-\alpha)^r+[k(1+\beta)+(\alpha+\beta)]^r\right\}^{-2}$$

$$+m(1-\alpha)^r(1+\beta)(k+1)\left\{r(\beta+1)[\left(k(1+\beta)+(\alpha+\beta)\right)]^{r-1}\right\}.$$

$$.\ \left\{m(1-\alpha)^r+[k(1+\beta)+(\alpha+\beta)]^r\right\}^{-2}$$

$$=m(1-\alpha)^r(1+\beta)[k(1+\beta)+\alpha+\beta]^r\left\{-[k(1+\beta)+(\alpha+\beta)]+r(1+\beta)(k+1)\right\}.$$

$$.\ [k(1+\beta)+\alpha+\beta]^{-1}\left\{m(1-\alpha)^r+[k(1+\beta)+(\alpha+\beta)]^r\right\}^{-2}$$

$$-m^2(1-\alpha)^{2r}(1+\beta)[k(1+\beta)+\alpha+\beta].[k(1+\beta)+\alpha+\beta]^{-1}.$$

$$.\ \left\{m(1-\alpha)^r+[k(1+\beta)+(\alpha+\beta)]^r\right\}^{-2}$$

$$=\frac{A(k)}{B(k)},$$

where

$$A(k)=m(1-\alpha)^r(1+\beta)[k(1+\beta)+\alpha+\beta]^r\{-[k(1+\beta)+(\alpha+\beta)]$$

$$+r(1+\beta)(k+1)\}-m^2(1-\alpha)^r(1+\beta)[k(1+\beta)+\alpha+\beta]\qquad (k\ge 1)$$

and using (2.13), then we have

$$A(k)=m^2(1-\alpha)^{2r}(1+\beta)\{r(1+\beta)(k+1)-2[k(1+\beta)+\alpha+\beta]\}\geq 0$$

for all $r\geq 2, 0\leq\alpha<1$ and $\beta\geq 0$. Then we have $g'(k)\geq 0$ for all $r\geq 2, 0\leq\alpha<1, \beta\geq 0$ and $k\geq 1$. Hence

$$\xi_m\leq 1-\frac{2m(\beta+1)(1-\alpha)^r}{m(1-\alpha)^r+(1+\alpha+2\beta)^r}.$$

By $(1+\alpha+2\beta)^r\geq m(1+2\beta)(1-\alpha)^r$, we can see that $0\leq\xi_m<1$. Thus the proof of Theorem 3 is completed.

Remark 4.

(i) Putting $\beta=0$ in Theorem 3, we have the result obtained by Aouf and Silverman [3, Theorem 3, with $n=1$];

(ii) Putting $\beta=0$ in Theorem 3, we obtain the result obtained by El-Ashwah et al. [7, Theorem 3, with $n=p=\gamma=1$] ;

(iii) Putting $r=2$ and $\alpha_j=\alpha$ $(j=1,2,...,m)$ in Theorem 3, we obtain the following corollary.

Corollary 3. Let the functions $f_j(z)(j=1,2,...,m)$ defined by (1.9), be in the class $U\Sigma S^*(\alpha,\beta)$ and let the function $F_m(z)$ be defined by

$$F_m(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m}(a_{k,j})^2\right)z^k \qquad (z\in U^*)$$

Then $F_m(z)\in U\Sigma S^*(\eta_m,\beta)(z\in U^*)$, where

$$\eta_m=1-\frac{2m(\beta+1)(1-\alpha)^2}{m(1-\alpha)^2+(1+\alpha+2\beta)^2}, \tag{2.21}$$

and

$$(1+\alpha+2\beta)^2\geq m(1+2\beta)(1-\alpha)^2.$$

The result is sharp, the extremal functions are

$$f_j(z)=\frac{1}{z}+\frac{(1-\alpha)}{(1+\alpha+2\beta)}z \quad (j=1,2,...,m).$$

Taking $m=2$ in Corollary 3, we obtain the following corollary.

Corollary 4. Let the functions $f_j(z)(j=1,2)$ defined by (1.9), be in the class $U\Sigma S^*(\alpha,\beta)$ and let the function $F_2(z)$ defined by

$$F_2(z)=\frac{1}{z}+\sum_{k=1}^{\infty}(a_{k,1}^2+a_{k,2}^2)z^k \quad (z\in U^*). \tag{2.22}$$

Then $F_2(z)\in U\Sigma S^*(\zeta_2,\beta)\left(z\in U^*\right)$, where

$$\zeta_2 = 1-\frac{4\left(\beta+1\right)\left(1-\alpha\right)^2}{2\left(1-\alpha\right)^2+(1+\alpha+2\beta)^2}, \tag{2.23}$$

and

$$\left(1+\alpha+2\beta\right)^2 \geq 2\left(1+2\beta\right)\left(1-\alpha\right)^2. \tag{2.24}$$

Remark 5.

(i) Putting $\beta=0$ in Corollary 4, we obtain the result obtained by Aouf and Silverman [3, Corollary 4, $n=1$].

(ii) Putting $\beta=0$ in Corollary 4, we obtain the result obtained by El-Ashwah et al. [7, Corollary 4, with $n=p=\gamma=1$].

Theorem 4. Let the functions $f_j\left(z\right)\left(j=1,2,...,m\right)$ defined by (1.9), be in the classes $U\Sigma K\left(\alpha_j,\beta\right)$ for each j, and let the function $F_m\left(z\right)$ defined by (2.11). Then $F_m\left(z\right)\in U\Sigma K\left(\gamma_m,\beta\right)$, where

$$\gamma_m \leq 1-\frac{2m\left(\beta+1\right)\left(1-\alpha\right)^r}{m\left(1-\alpha\right)^r+(1+\alpha+2\beta)^r}\quad, \tag{2.25}$$

$$\alpha = \min_{1\leq j\leq m}\{\alpha_j\}$$

and

$$(1+\alpha+2\beta)^r \geq m\left(1+2\beta\right)\left(1-\alpha\right)^r. \tag{2.26}$$

Proof. Since $f_j\left(z\right)\in U\Sigma K\left(\alpha_j,\beta\right)$, using Lemma 2, we obtain

$$\sum_{k=1}^{\infty}\frac{k\left[k\left(1+\beta\right)+\left(\alpha_j+\beta\right)\right]}{(1-\alpha_j)}a_{k,j}\leq 1 \qquad \left(j=1,2\right). \tag{2.27}$$

Thus, the proof of Theorem 4 is similar to that of Theorem 3, where Lemma 2 is used instead of Lemma 1.

By taking $r=2$ and $\alpha_j=\alpha\left(j=1,2,...,m\right)$ in Theorem 4, we obtain the following corollary.

Corollary 5. Let the functions $f_j\left(z\right)\left(j=1,2,...,m\right)$ defined by (1.9) be in the class $U\Sigma K\left(\alpha,\beta\right)$ and let the function $F_m\left(z\right)$ defined by

$$F_m(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m}(a_{k,j})^2\right)z^k \qquad \left(z\in U^*\right). \tag{2.28}$$

Then $F_m(z)\in U\Sigma K\left(\zeta_m,\beta\right)$, where

$$\zeta_m = 1 - \frac{2m(\beta+1)(1-\alpha)^2}{m(1-\alpha)^2 + (1+\alpha+2\beta)^2}, \tag{2.29}$$

and

$$(1+\alpha+2\beta)^2 \geq m(1+2\beta)(1-\alpha)^2. \tag{2.30}$$

Taking $m = 2$ in Corollary 5, we obtain the following corollary.

Corollary 6. Let the functions $f_j(z)(j = 1,2)$ defined by (1.9) be in the class $U\Sigma K(\alpha,\beta)$, where α satisfies (2.24) and let the function $F_2(z)$ defined by (2.22). Then $F_2(z) \in U\Sigma K(\zeta_2,\beta)(z \in U^*)$, where ζ_2 is defined by (2.23).

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