Some Classes of Univalent Analytic Functions Involving Jung, Kim and Srivastava Operator

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Abstract

This paper aimed to investigate and study some new subclasses of univalent starlike, convex and close – to – convex functions which defined by Jung, Kim and Srivastava operator P^{α}. The inclusion relations are established and the properties of integral operator P^{α} in these subclasses is discussed.

Also We obtain some results of inclusion relations for composition operation of the operator P^{α}, and Bernardi integral operator $J_{c,1}$: $P^{\alpha}(J_{c,1}f(z))$ of functions f(z), belonging to these subclasses of univalent analytic functions.

The main results of this paper are to obtain the properties and inclusion relations for subclasses of univalent functions involving the operator P^{α} and $P^{\alpha}(J_{c,1}f(z))$.

Key wards: Univalent Analytic Functions ; starlike functions; Convex functions; Close – to – convex functions; Jung , Kim and Srivastava Operator.

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1. Introduction

Let A denote the class of function of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \qquad (1.1)$$

Which are analytic in the open unit disk $U = \{z: |z| < 1\}$.

A function $f \in A$ is said to be in the class $S^*(\alpha)$ of univalent starlike functions of order α , if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha (0 \leq \alpha < 1 , z \in U) . \qquad (1.2)$$

We write $S^*(0) = S^*$ the class of univalent starlike functions in U.

A function $f \in A$ is said to be in the class C(α) of univalent convex functions of order α, if it satisfies

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \leq \alpha < 1, z \in U). \qquad (1.3)$$

The class of univalent convex functions in U is denoted by $C = C(0)$. It follows from (1.2) and (1.3) that

$f(z) \in C(\alpha)$ if and only if

$$zf'(z) \in S^*(\alpha), (0 \leq \alpha < 1). \qquad (1.4)$$

These classes S^* and C was introduced by Goodman [1, 2].

Furthermore, a function $f \in A$ is said to be univalent close-to-convex of order β and Type γ in U , if there exists a function $g \in S^*(\gamma)$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > \beta, \ (0 \leq \beta , \gamma < 1 , z \in U) . \qquad (1.5)$$

We denote by B(β, γ) , the subclass of A consisting of all such functions.

The class B(β, γ), was studied by Kaplan [3] and Libera [4].

For the functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} \ z^k. \qquad (1.6)$$

We denote the Hadamard product (or convolution)

$$f_1(z) * f_2(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k \ . \qquad (1.7)$$

For $f(z) \in A$ given by (1.1), Jung, Kim and Srivastava [5], have introduced the following one parameter integral operator:

$P^{\alpha}f(z) = \frac{2^{\alpha}}{z\Gamma(\alpha)}\int_0^z \left(ln\frac{z}{t}\right)^{\alpha-1} f(t)dt$, $(\alpha > 0)$, (1.8)

or equivalently by

$$P^{\alpha}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\alpha} a_n z^n \ , (\alpha > 0). \qquad (1.9)$$

From (1.9), we get the identity

$z\left(P^{\alpha}f(z)\right)' = 2P^{\alpha-1}f(z) - P^{\alpha}f(z)$. (1.10)

Using the operator $P^{\alpha}f(z)$, we now introduce the following classes.

$$S_{\alpha}^{*}(\lambda) = \{f \in A : P^{\alpha-1}f(z) \in S^{*}(\lambda), 0 \le \lambda < 1, \alpha > 0\}$$

$$C_{\alpha}(\lambda) = \{f \in A : P^{\alpha-1}f(z) \in C(\lambda), 0 \le \lambda < 1, \alpha > 0\}$$

and

$$B_{\alpha}(\beta,\gamma) = \{f \in A : P^{\alpha-1}f(z) \in B(\beta,\gamma), 0 \le \beta, \gamma < 1, \alpha > 0\}.$$

In this paper, we shall establish inclusion relations for these classes and investigate integral operator in these classes.

2. Inclusion Relations

In order to prove our main results, we shall require the following lemma.

Lemma [6,7,8]

Let $\phi : D \longrightarrow \mathbb{C}$, $D \subset \mathbb{C}\times\mathbb{C}$ $(\mathbb{C}\ is\ thecomplexplane)$.

and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) $\phi(u,v)$is continuous in D; (ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$; (iii) for all $(iu_2, v_1) \in D$ such that

$$v_1 \le -\frac{(1+u_2^2)}{2}, \qquad Re\{\phi(iu_2, v_1)\} \le 0.$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in the unit disk U, such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $Re\left\{\phi\left(p(z), zp'(z)\right)\right\} > 0$ $(z \in U)$.

Then $Re\{p(z)\} > 0$ $(z \in U)$.

Our first theorem is stated as:

Theorem 1.

$S_{\alpha}^{*}(\lambda) \subset S_{\alpha+1}^{*}(\lambda)$, $(\alpha > 0)$.

Proof. Let $f \in S_{\alpha}^{*}(\lambda)$ and set

$\frac{z\left(P^{\alpha}f(z)\right)'}{P^{\alpha}f(z)} = \lambda + (1-\lambda)h(z)$, (2.1)

where$h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ using the identity (1.10), we have

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}=\frac{1}{2}\left[\frac{z\left(P^{\alpha}f(z)\right)'}{P^{\alpha}f(z)}+1\right]=\frac{1}{2}[\lambda+(1-\lambda)h(z)+1]. \qquad (2.2)$$

Differentiating (2.2), and multiply both sides by z, we obtain

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}f(z)}=\frac{z\left(P^{\alpha}f(z)\right)'}{P^{\alpha}f(z)}+\frac{(1-\lambda)zh'(z)}{\lambda+(1-\lambda)h(z)+1}$$

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}f(z)}-\lambda=(1-\lambda)h(z)+\frac{(1-\lambda)zh'(z)}{1+\lambda+(1-\lambda)h(z)}. \qquad (2.3)$$

Now we form the function $\phi(u,v)$ by taking $u=h(z), v=zh'(z) in\ (2.3) as$:

$$\phi(u,v)=(1-\lambda)u+\frac{(1-\lambda)v}{1+\lambda+(1-\lambda)u}. \qquad (2.4)$$

It is easy to see that the function $\phi(u,v)$ satisfies conditions (i) and (ii) of Lemma 1, in $D=\left(\mathbb{C}-\left\{\frac{1+\lambda}{\lambda-1}\right\}\right)\times\mathbb{C}$.

To verify condition (iii), we calculate as follows:

$$Re\ \phi(iu_2,v_1)=Re\left\{\frac{(1-\lambda)v_1}{1+\lambda+(1-\lambda)iu_2}\right\}$$

$$=\frac{(1-\lambda)(1+\lambda)v_1}{(1+\lambda)^2+(1-\lambda)^2u_2^2}$$

$$\leq-\frac{(1-\lambda)(1+\lambda)(1+u_2^2)}{2\{(1+\lambda)^2+(1-\lambda)^2u_2^2\}}\leq 0,$$

where $v_1\leq-\frac{(1+u_2^2)}{2}$ and $(iu_2,v_1)\in D$. Therefore the function $\phi(u,v)$ satisfies the conditions of Lemma 1. This shows that if

$Re\ \phi\left(h(z),zh'(z)\right)>0\ \ (z\in U)$, then $Re h(z)>0\ (z\in U)$, that is if $f(z)\in S_\alpha^*(\lambda)$ then $f(z)\in S_{\alpha+1}^*(\lambda)$. The proof is complete.

Theorem 2.

$C_\alpha(\lambda)\subset C_{\alpha+1}(\lambda)\,,(\alpha>0)$.

Proof. $f(z)\in C_\alpha(\lambda)\Leftrightarrow P^{\alpha-1}f(z)\in C(\lambda)$

$$\Leftrightarrow z\left(P^{\alpha-1}f(z)\right)'\in S^*(\lambda)$$

$$\Leftrightarrow P^{\alpha-1}(zf')\in S^*(\lambda)$$

$$\Leftrightarrow zf'(z)\in S_\alpha^*(\lambda)$$

$$\Rightarrow zf'(z)\in S_{\alpha+1}^*(\lambda)$$

$$\Leftrightarrow P^{\alpha}\left(zf'(z)\right)\in S^*(\lambda)$$

$$\Leftrightarrow z\left(P^{\alpha}f(z)\right)' \in S^{*}(\lambda)$$
$$\Leftrightarrow P^{\alpha}f(z) \in C(\lambda)$$
$$\Leftrightarrow f(z) \in C_{\alpha+1}(\lambda)$$

Which evidently proves Theorem 2.

Theorem 3.

$B_{\alpha}(\beta,\gamma) \subset B_{\alpha+1}(\beta,\gamma)$.

Proof .Let $f(z) \in B_{\alpha}(\beta,\gamma)$. then there exists a function $k(z) \in S^{*}(\gamma)$ such that

$$Re\left\{\frac{z\left(P^{\alpha-1}f(z)\right)'}{k(z)}\right\} > \beta\ ,(z \in U).$$

Taking the function $g(z)$ which satisfies $P^{\alpha-1}g(z) = k(z)$, we have $g(z) \in S_{\alpha}^{*}(\gamma)$and

$$Re\left\{\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}g(z)}\right\} > \beta\ ,(z \in U).$$

Now set

$$\frac{z\left(P^{\alpha}f(z)\right)'}{P^{\alpha}g(z)} = \beta + (1-\beta)h(z), \qquad (2.5)$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ from (2.5), we have

$$z\left(P^{\alpha}f(z)\right)' = P^{\alpha}g(z)[\beta + (1-\beta)h(z)]. \qquad (2.6)$$

So that from (2.6) and the identity (1.10), we have

$$2z\left(P^{\alpha-1}f(z)\right)' = z\left(P^{\alpha}g(z)\right)'[\beta + (1-\beta)h(z)]$$
$$+P^{\alpha}g(z)[(1-\beta)zh'(z)] + z\left(P^{\alpha}f(z)\right)'. \qquad (2.7)$$

Now apply (1.10) for the function $g(z)$ and use (2.7) to obtain

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}g(z)} = \beta + (1-\beta)h(z) + \frac{P^{\alpha}g(z)}{P^{\alpha-1}g(z)}\left(\frac{(1-\beta)zh'(z)}{2}\right). \quad (2.8)$$

Since $g(z) \in S_{\alpha}^{*}(\gamma)$ and $S_{\alpha}^{*}(\gamma) \subset S_{\alpha+1}^{*}(\gamma)$, we let

$$\frac{z\left(P^{\alpha}g(z)\right)'}{P^{\alpha}g(z)} = \alpha + (1-\alpha)H(z)\ ,$$

where$Re\left(H(z)\right) > 0\ ,\ \ (z \in U)$. Thus(2.8), can bewritten as :

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{1+\gamma+(1-\gamma)H(z)}. \ (2.9)$$

Now we form the function $\phi(u,v)$ by taking
$u = h(z), v = zh'(z) in\ (2.9) as$:

$$\phi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{1+\gamma+(1-\gamma)H(z)}.$$

It is easy to see that the function $\phi(u,v)$ satisfies conditions (i) and (ii) of Lemma 1, in $D = \mathbb{C} \times \mathbb{C}$.

To verify condition (iii), we proceed as follows:

$$Re\phi(iu_2, v_1) = \frac{(1-\beta)v_1[1+\gamma+(1-\gamma)h_1(x,y)]}{[1+\gamma+(1-\gamma)h_1(x,y)]^2 + [(1-\gamma)h_2(x,y)]^2},$$

where $H(z) = h_1(x,y) + ih_2(x,y)$, $h_1(x,y)$ *and* $h_2(x,y)$ being functions of x and y and

$$ReH(z) = h_1(x,y) > 0.$$

By putting $v_1 \le \frac{-(1+u_2^2)}{2}$, we have

$$Re\phi(iu_2, v_1) = -\frac{(1-\beta)(1+u_2^2)[1+\gamma+(1-\gamma)h_1(x,y)]}{[1+\gamma+(1-\gamma)h_1(x,y)]^2 + [(1-\gamma)h_2(x,y)]^2} < 0.$$

Hence $Reh(z) > 0$, $(z \in U)$, and so $f(z) \in \beta_{\alpha+1}(\beta,\gamma)$.

This completes the proof of Theorem 3.

3. Integral operator

For $c > -1$ and $f(z) \in A$, Bernardi [9],was introduced the integral operator $J_{c,1}f(z)$ as:

$J_{c,1}\big(f(z)\big) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt.$ (3.1)

In particular, the operator $J_{1,1}$ was studied earlier by Libera [10] and Livingston [11].

Theorem 4.

Let $C > -\gamma$ If $f(z) \in S_\alpha^*(\gamma)$,then $J_{c,1}f(z) \in S_\alpha^*(\gamma)$.

Proof: Let $f(z) \in S_\alpha^*(\gamma)$, from (3.1), we have

$$z\left(P^{\alpha-1}J_{c,1}f(z)\right)' = (c+1)P^{\alpha-1}f(z) - cP^{\alpha-1}J_{c,1}f(z)\,. \qquad (3.2)$$

Set

$$\frac{z\left(P^{\alpha-1}J_{c,1}f(z)\right)'}{P^{\alpha-1}J_{c,1}f(z)} = \gamma + (1-\gamma)h(z), \qquad (3.3)$$

where $h(z) = 1 + c_1z + c_2z^2 + \cdots$

Using the identity (3.3), (3.2), we have

$\frac{P^{\alpha-1}f(z)}{P^{\alpha-1}J_{c,1}f(z)} = \frac{1}{c+1}[c+\gamma+(1-\gamma)h(z)]$. (3.4)

Differentiating (3.4),and multiply both sides by z, we get

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}f(z)} = \frac{z\left(P^{\alpha-1}J_{c,1}f(z)\right)'}{P^{\alpha-1}J_{c,1}f(z)} + \frac{(1-\gamma)zh'(z)}{c+\gamma+(1-\gamma)h(z)}.$$

From (3.3), we get

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}f(z)} - \gamma = (1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{c+\gamma+(1-\gamma)h(z)}. \qquad (3.5)$$

Now we form the function $\phi(u,v)$ by taking

$u = h(z), v = zh'(z) in\ (3.5) as$

$$\phi(u,v) = (1-\gamma)u + \frac{(1-\gamma)v}{c+\gamma+(1-\gamma)u}.$$

It is easy to see that the function $\phi(u,v)$ satisfies conditions (i) and (ii) of Lemma1, in $D = \left(\mathbb{C} - \left\{\frac{c+\gamma}{\gamma-1}\right\}\right) \times \mathbb{C}$.

To verify condition (iii), we calculate as follows:

$$Re\phi(iu_2, v_1) = Re\left\{\frac{(1-\gamma)v_1}{c+\gamma+(1-\gamma)iu_2}\right\}$$

$$= \frac{(1-\gamma)(c+\gamma)v_1}{(c+\gamma)^2+(1-\gamma)^2u_2^2}$$

$$\leq -\frac{(1-\gamma)(c+\gamma)(1+u_2^2)}{2(c+\gamma)^2+(1-\gamma)^2u_2^2} < 0,$$

where $v_1 \leq -\frac{(1+u_2^2)}{2}$, and $(iu_2, v_1) \in D$, therefore the function$\phi(u,v)$, satisfies conditions of Lemma 1. This shows that if $Re\phi(u,v) > 0$, $z \in U$, then $Re(h(z)) > 0$ then $J_{c,1}f(z) \in S_\alpha^*(\gamma)$. The proof is complete.

Theorem 5.

Let $> -\gamma$, If $f(z) \in c_\alpha(\gamma)$, then $J_{c,1}f(z) \in C_\alpha(\gamma)$.

Proof. Let $f(z) \in c_\alpha(\gamma) \Longrightarrow zf'(z) \in S_\alpha^*(\gamma)$

$$\Longrightarrow J_{c,1}\left(zf'(z)\right) \in S_\alpha^*(\gamma)\ [Theroem\ 4]$$

$$zJ_{c,1}\left(f'(z)\right) \in S_\alpha^*(\gamma) \Leftrightarrow z\left(J_{c,1}f(z)\right)' \in S_\alpha^*(\gamma)$$

$$\Leftrightarrow J_{c,1}f(z) \in c_\alpha(\gamma).$$

The proof is complete

Theorem 6.

Let $c > -\gamma$. If $f(z) \in B_\alpha(\beta,\gamma)$ then $J_{c,1}f(z) \in B_\alpha(\beta,\gamma)$.

Proof. Let $f(z) \in B_\alpha(\beta,\gamma)$. Then by the definition there exists a function $g(z) \in S_\alpha^*(\gamma)$ such that

$$Re\left\{\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}g(z)}\right\} > \beta\,, \quad (z \in U).$$

Put

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}J_{c,1}g(z)} = \beta + (1-\beta)h(z), (3.6)$$

where$h(z) = 1 + c_1 z + c_2 z^2 + \cdots$From (3.2), we have

$$\frac{(c+1)P^{\alpha-1}f(z) - cP^{\alpha-1}J_{c,1}f(z)}{P^{\alpha-1}J_{c,1}g(z)} = \beta + (1-\beta)h(z)$$

$$(c+1)P^{\alpha-1}f(z) - cP^{\alpha-1}J_{c,1}f(z) = \left(P^{\alpha-1}J_{c,1}g(z)\right)\left(\beta + (1-\beta)h(z)\right).$$

Differentiating both sides and multiply by z, we get

$$(c+1)z\left(P^{\alpha-1}f(z)\right)' = z\left(P^{\alpha-1}J_{c,1}\left(g(z)\right)\right)'[\beta + (1-\beta)h(z)] + \left(P^{\alpha-1}J_{c,1}\left(g(z)\right)\right)\left((1-\beta)zh'(z)\right) + cz\left(P^{\alpha-1}J_{c,1}\left(f(z)\right)\right)'. \qquad (3.7)$$

Now apply (3.2) for the function g(z) and use (3.7), to obtain

$$\frac{z\left(P^{\alpha-1}f(z)\right)'}{P^{\alpha-1}g(z)} = \beta + (1-\beta)h(z) + \frac{P^{\alpha-1}J_{C,1}(g(z))}{P^{\alpha-1}g(z)}.\frac{(1-\beta)zh'(z)}{c+1}\,. \qquad (3.8)$$

Since $g(z) \in s_\alpha^*(\gamma)$, then from Theorem 4,

$J_{c,1}(f(z)) \in S_\alpha^*(\gamma)$,we let

$$\frac{z\left(P^{\alpha-1}J_{C,1}(g(z))\right)'}{P^{\alpha-1}g(z)} = \gamma + (1-\gamma)H(z).$$

Where $Re\left(H(z)\right) > 0$ $(z \in U)$. Thus (3.8) can be written as:

$$\frac{z\left(P^{\alpha-1}f(Z)\right)'}{P^{\alpha-1}g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{c+\gamma+(1+\gamma)H(z)}. \qquad (3.9)$$

Now we form the function $\emptyset(u,v)$ by taking $u = h(z), v = zh'(z)\ in\ (3.9) as$:

$$\phi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{u+\gamma+(1-\gamma)H(z)}.$$

It is easy to see that the function $\phi(u,v)$satisfies conditions (i) and (ii) of Lemma1, in $D = \mathbb{C} \times \mathbb{C}$.

To verify condition (iii) we proceed as follows:

$$Re\phi(iu_2, v_1) = \frac{(1-\beta)v_1[1+\gamma+(1-\gamma)h_1(x,y)]}{[1+\gamma+(1-\gamma)h_1(x,y)]^2+[(1-\gamma)h_2(x,y)]^2},$$

where$H(z) = h_1(x,y) + ih_2(x,y)$, $h_1(x,y)$ and $h_2(x,y)$ being functions of x and y and $ReH(z) = h_1(x,y) > 0$.

By putting $v_1 \leq -\frac{(1+u_2^2)}{2}$, we have

$$Re\phi(iu_2, v_1) = -\frac{(1-\beta)(1+u_2^2)[1+\gamma+(1-\gamma)h_1(x,y)]}{2\{[1+\gamma+(1-\gamma)h_1(x,y)]^2+[(1-\gamma)h_2(x,y)]^2\}} \leq 0.$$

Hence $Reh(z) > 0$, $(z \in U)$

$and\ so\ f(z) \in \beta_{\alpha+1}(\beta,\gamma)$. This completes the proof of Theorem 6.

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