Solitons and other exact solutions for two nonlinear PDEs in mathematical physics using the generalized (G'/G)-expansion method

Khaled A. E. Alurr [1], *Ragab M. A. Almasroub* [2], *Blied S. Dalem* [3] *and A. M. H. Arwiniya* [4]

[1]Department of Mathematics, Faculty of Science,
Elmergib University, Khoms, Libya.

[2]Department of Mathematics, Faculty of Science,
Azzyatuna University, Tarhuna, Libya.

[3]Department of Physics, Faculty of Arts & Science,
Elmergib University, Msallata, Libya.

[4]Department of Mathematics, Faculty of Science,
Alasmarya Islamic University, Zliten, Libya.

Abstract

In this article, we apply the generalized (G'/G)-expansion method with the aid of computer algebra systems (CAS) such as Maple or Mathematica to construct many new types of Jacobi elliptic function solutions for two nonlinear partial differential equations (PDEs) describing the nonlinear low-pass electrical lines and pulse narrowing nonlinear transmission lines. Based on Kirchhoff's law, the given nonlinear PDEs have been derived and can be reduced to nonlinear ordinary differential equations (ODEs) using a simple transformation. Soliton wave solutions or periodic function solutions are obtained from the Jacobi elliptic function solutions when the modulus of the Jacobi elliptic functions approaches to one or zero respectively. Comparing our new results with the well-known results are given. The used method in this article is straightforward, concise and it can also be applied to other nonlinear PDEs in mathematical physics.

Keywords: Generalized (G'/G)-expansion method; Exact solutions; Nonlinear low-pass electrical lines; Pulse narrowing nonlinear transmission lines.

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1. Introduction

In the recent years, investigations of exact solutions to nonlinear PDEs play an important role in the study of nonlinear physical phenomena in such as fluid mechanics, hydrodynamics, optics, plasma physics, solid state physics, biology and so on. Several methods for finding the exact solutions to nonlinear equations in mathematical physics have been presented, such as the inverse scattering method [1], the Hirota bilinear transform method [2], the $(G\prime/G)$-expansion method [3-6], the generalized $(G\prime/G)$-expansion method [7-9], the new mapping method [10-12], the Bäcklund transform method [13-15], the generalized projective Riccati equations method [16-18], the Jacobi elliptic function expansion method [19-21], the new Jacobi elliptic function expansion method [22-24], the $(\frac{G'}{G}, \frac{1}{G})$-expansion method [24-29], the extended auxiliary equation method [30,31] and so on.

The objective of this article is to use the generalized $\left(\frac{G'}{G}\right)$-expansion method to construct many exact solutions including Jacobi elliptic function solutions and solitons wave solutions of the following nonlinear PDEs:

(I) The nonlinear PDE governing wave propagation in nonlinear low-passe lectrical transmission lines [23,32]:

$$\frac{\partial^2 V(x,t)}{\partial t^2} - \alpha\frac{\partial^2 V^2(x,t)}{\partial t^2} + \beta\frac{\partial^2 V^3(x,t)}{\partial t^2} - \delta^2\frac{\partial^2 V(x,t)}{\partial x^2} - \frac{\delta^4}{12}\frac{\partial^4 V(x,t)}{\partial x^4} = 0, \tag{1.1}$$

where α, β and δ are constants, while $V(x,t)$ is the voltage in the transmission lines. The variable x is interpreted as the propagation distance and t is the slow time. The physical details of the derivation of Eq. (1.1) using the Kirchhoff's laws given in [32], which are omitted here for simplicity. Note that Eq. (1.1) has been discussed in [32] using an auxiliary equation method listed in [33] and its exact solutions have been found. Also, Eq. (1.1) has been discussed in [23] using a new Jacobi elliptic function expansion method.

(II) The nonlinear PDE describing pulse narrowing nonlinear transmission lines [24]:

$$\frac{\partial^2 E(x,t)}{\partial t^2} - \frac{1}{LC_0}\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{b_1}{2}\frac{\partial^2 E^2(x,t)}{\partial t^2} - \frac{\sigma^2}{12LC_0}\frac{\partial^4 E(x,t)}{\partial x^4} = 0, \tag{1.2}$$

where $E(x,t)$ is the voltage of the pulse and C_0, L, σ and b_1 are constants. The physical details of the derivation of Eq. (1.2) is elaborated in [34] using the Kirchhoff's current law and Kirchhoff's voltage law, which are omitted here for simplicity. Eq. (1.2) has been discussed in [24] using a new Jacobi elliptic function expansion method and its exact solutions have been found.

This article is organized as follows: In Sec. 2, the description of the generalized $\left(\frac{G'}{G}\right)$-expansion method is given. In Sec. 3, we use the given method described in Sec. 2, to find exact solutions of Eqs. (1.1) and (1.2). In Sec. 4, physical explanations of some results are presented. In Sec. 5, some conclusions are obtained.

2. Description of the generalized $\left(\frac{G'}{G}\right)$-expansion method

Consider a nonlinear PDE in the form:

$$P(V, V_x, V_t, V_{xx}, V_{tt}, ...) = 0, \tag{2.1}$$

where $V = V(x,t)$ is a unknown function, P is a polynomial in $V(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. Let us now give the main steps of the generalized $\left(\frac{G'}{G}\right)$-expansion method [7-9]:

Step 1. We look for the voltage $V(x,t)$ in the traveling form:

$$V(x,t) = V(\xi), \qquad \xi = \sqrt{k}(x - \omega t), \tag{2.2}$$

where k and ω are undetermined positive parameters, and ω is the velocity of propagation, to reduce Eq. (2.1) to the following nonlinear ODE:

$$H(V, V', V'', ...) = 0, \tag{2.3}$$

where H is a polynomial of $V(\xi)$ and its total derivatives $V'(\xi), V''(\xi), ...$ and $' = \dfrac{d}{d\xi}$.

Step 2. We assume that the solution of Eq. (2.3) has the form:

$$V(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i, \tag{2.4}$$

where $a_i \ \ (i = 1, 2, ..., N)$ are constants to be determined later, provided $a_N \neq 0$, and $G = G(\xi)$ satisfies the following Jacobi elliptic equation

$$G'^2(\xi) = R + QG^2(\xi) + PG^4(\xi), \tag{2.5}$$

where $R,\ Q$ and P are constants.
Step 3. We determine the positive integer N in (2.4) by balancing the highest-order derivatives and the highest nonlinear terms in Eq. (2.3).

Step 4. Substituting (2.4) along with Eqs. (2.5) into Eq. (2.3) and collecting all the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$ $(i = 0, 1, 2, ...)$, then setting these coefficients to zero, yield a set of algebraic equations, which can be solved by using the CAS to find the values of a_i , $R,\ Q$ and P.

Step 5. It is well-known that Eqs. (2.5) have many families of solutions listed in [7-9].

Step 6. Substituting the values of a_i , R, Q and P as well as the solutions of Step 5, into (2.4) we have the exact solutions of Eq. (2.1).

3. Applications

In this section, we apply the generalized $\left(\frac{G'}{G}\right)$ -expansion method of Sec. 2 to find new exact solutions of Eqs. (1.1) and (1.2) in the following subsections:

3.1. Exact wave solutions of Eq. (1.1) using the method of Sec. 2

In order to solve Eq. (1.1), we use the transformation (2.2) to reduce Eq. (1.1) to the following nonlinear ODE:

$$\frac{d^2}{d\xi^2}\left\{\frac{k^2\delta^4}{12}\frac{d^2V}{d\xi^2}+(k\delta^2-k\omega^2)V+\alpha k\omega^2V^2-\beta k\omega^2V^3\right\}=0. \tag{3.1.1}$$

Integrating Eq. (3.1.1) twice and vanishing the constants of integration, we find the following ODE:

$$\frac{K^2}{12}\frac{d^2V}{d\xi^2}+(K-U)V+\alpha UV^2-\beta UV^3=0. \tag{3.1.2}$$

where $K=k\delta^2$ and $U=k\omega^2$.
Balancing $\frac{d^2V}{d\xi^2}$ with V^3 gives $N=1$. Therefore, (2.4) reduces to

$$V(\xi)=a_0+a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \tag{3.1.3}$$

where a_0 and a_1 are constants to be determined provided that $a_1\neq 0$.
Now, substituting (3.1.3) along with the Jacobi elliptic equation (2.5) into Eq. (3.1.2) and collecting all the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$, $(i=0,1,2,3)$ and setting them to be zero, we have the following algebraic equations:

$\left(\frac{G'(\xi)}{G(\xi)}\right)^3:\ \frac{1}{6}K^2a_1-\beta Ua_1^3=0,$

$\left(\frac{G'(\xi)}{G(\xi)}\right)^2:\ -3\beta Ua_0a_1^2+\alpha Ua_1^2=0,$

$\left(\frac{G'(\xi)}{G(\xi)}\right):\ \frac{-1}{6}K^2a_1Q+(K-U)a_1+2\alpha Ua_0a_1-3\beta Ua_0^2a_1=0,$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0:\ (K-U)a_0+\alpha Ua_0^2-\beta Ua_0^3=0. \tag{3.1.4}$$

On solving the above algebraic equations (3.1.4) using the CAS, we have the following result:

$$K=-\frac{6\alpha^2}{Q(2\alpha^2-9\beta)},\quad U=\frac{54\alpha^2\beta}{Q(2\alpha^2-9\beta)^2},\quad a_0=\frac{\alpha}{3\beta},\quad a_1=\pm\frac{\alpha}{3\beta\sqrt{Q}},\quad P=P\ ,\quad R=R\ , \tag{3.1.5}$$

where $Q>0$.

Substituting (3.1.5) into (3.1.3) yields

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{1}{\sqrt{Q}}\left(\frac{G'(\xi)}{G(\xi)}\right)\right], \tag{3.1.6}$$

where $\xi=\sqrt{-\frac{6\alpha^2}{Q(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{54\alpha^2\beta}{Q(2\alpha^2-9\beta)^2}}t$, $2\alpha^2<9\beta$, $\beta>0$.

With reference to solving Eq. (2.5), we deduce that the Jacobi elliptic function solutions and other exact solutions of Eq. (1.1) as follows:

Case 1. Choosing $P=-m^2, Q=2m^2-1, R=1-m^2$ and $G(\xi)=\mathrm{cn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\mathrm{sc}(\xi)\,\mathrm{dn}(\xi)}{\sqrt{2m^2-1}}\right], \tag{3.1.7}$$

where $\xi=\sqrt{-\frac{6\alpha^2}{(2m^2-1)(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{54\alpha^2\beta}{(2m^2-1)(2\alpha^2-9\beta)^2}}t$.
If $m\to1$, then Eq. (1.1) has the kink soliton wave solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\tanh(\xi)\right] \tag{3.1.8}$$

where $\xi=\sqrt{-\frac{6\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{54\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t$.
Case 2. Choosing $P=-1, Q=2-m^2, R=m^2-1$ and $G(\xi)=\mathrm{dn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\mathrm{ds}(\xi)}{\sqrt{2-m^2}\,\mathrm{cn}(\xi)}\right], \tag{3.1.9}$$

where $\xi=\sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t$.
If $m\to1$, then we have the same kink soliton wave solutions Eq. (3.1.8).

Case 3. Choosing $P=1-m^2, Q=2-m^2, R=1$ and $G(\xi)=\mathrm{sc}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\mathrm{ds}(\xi)}{\mathrm{cn}(\xi)\sqrt{2-m^2}}\right], \tag{3.1.10}$$

where $\xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t$.
If $m \to 0$, then Eq. (1.1) has the exact wave solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\sec(\xi)\csc(\xi)}{\sqrt{2}}\right], \tag{3.1.11}$$

where $\xi = \sqrt{-\frac{3\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{27\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t$.

If $m \to 1$, then Eq. (1.1) has the anti-kink soliton wave solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \coth(\xi)\right], \tag{3.1.12}$$

where $\xi = \sqrt{-\frac{6\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t$.
Case 4. Choosing $P = \frac{1}{4}, Q = \frac{1-2m^2}{2}, R = \frac{1}{4}$ and $G(\xi) = \mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \mp \frac{\sqrt{2}\,\mathrm{ds}(\xi)}{\sqrt{1-2m^2}}\right], \tag{3.1.13}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(1-2m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(1-2m^2)(2\alpha^2-9\beta)^2}}t$.
If $m \to 0$, then Eq. (1.1) has the exact wave solutions

$$V(\xi) = \frac{\alpha}{3\beta}[1 \pm \sqrt{2}\csc(\xi)], \tag{3.1.14}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t$.

Case 5. Choosing $P = 1-m^2, Q = 2-m^2, R = 1$ and $G(\xi) = \mathrm{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\mathrm{nc}(\xi)\,\mathrm{ds}(\xi)}{\sqrt{2-m^2}}\right], \tag{3.1.15}$$

where $\xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t$.
If $m \to 0$, then we have the same exact solutions (3.1.11).
If $m \to 1$, then we have the same anti-kink soliton wave solutions (3.1.12).

Case 6. Choosing $P = m^2(m^2-1), Q = 2m^2-1, R = 1$ and $G(\xi) = \mathrm{ds}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\mathrm{cd}(\xi)\,\mathrm{ns}(\xi)}{\sqrt{2m^2-1}}\right], \tag{3.1.16}$$

where $\xi = \sqrt{-\frac{6\alpha^2}{(2m^2-1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2m^2-1)(2\alpha^2-9\beta)^2}}t$.
If $m \to 1$, then we have the same ant-kink soliton wave solutions (3.1.12).

Case 7. Choosing $P = m^2-1, Q = 2-m^2, R = -1$ and $G(\xi) = \mathrm{nd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{m^2\,\mathrm{cd}(\xi)\,\mathrm{sd}(\xi)\,\mathrm{dn}(\xi)}{\sqrt{2-m^2}}\right], \tag{3.1.17}$$

where $\xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t$.

If $m \to 1$, then we have the same kink soliton wave solutions (3.1.8).

Case 8. Choosing $P = \frac{m^2-1}{4}, Q = \frac{m^2+1}{2}, R = \frac{m^2-1}{4}$ and $G(\xi) = m\,\mathrm{sd}(\xi) \pm \mathrm{nd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{m\sqrt{2}\,\mathrm{cd}(\xi)}{\sqrt{1+m^2}}\right], \tag{3.1.18}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(1+m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(1+m^2)(2\alpha^2-9\beta)^2}}t$.

If $m \to 1$, then we have the trivel solution.

Case 9. Choosing $P = \frac{1}{4}, Q = \frac{1-2m^2}{2}, R = \frac{1}{4}$ and $G(\xi) = \mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\sqrt{2}\,\mathrm{ds}(\xi)}{\sqrt{1-2m^2}}\right], \tag{3.1.19}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(1-2m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(1-2m^2)(2\alpha^2-9\beta)^2}}t$.

If $m \to 0$, then we have the same exact wave solutions (3.1.14).

Case 10. Choosing $P = \frac{1-m^2}{4}, Q = \frac{m^2+1}{2}, R = \frac{1-m^2}{4}$ and $G(\xi) = \mathrm{nc}(\xi) \pm \mathrm{sc}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\sqrt{2}\,\mathrm{dc}(\xi)}{\sqrt{m^2+1}}\right], \tag{3.1.20}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(m^2+1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(m^2+1)(2\alpha^2-9\beta)^2}}t.$

If $m \to 0$, then Eq. (1.1) has the exact wave solutions

$$V(\xi) = \frac{\alpha}{3\beta}[1 \pm \sqrt{2}\sec(\xi)], \tag{3.1.21}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t.$
If $m \to 1$, then we have the trivel solution.

Case 11. Choosing $P = \frac{1}{4}, Q = \frac{m^2-2}{2}, R = \frac{m^2}{4}$ and $G(\xi) = \mathrm{ns}(\xi) \pm \mathrm{ds}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\sqrt{2}\,\mathrm{cs}(\xi)}{\sqrt{m^2-2}}\right], \tag{3.1.22}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(m^2-2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(m^2-2)(2\alpha^2-9\beta)^2}}t.$

Case 12. Choosing $P = \frac{m^2}{4}, Q = \frac{m^2-2}{2}, R = \frac{m^2}{4}$ and $G(\xi) = \sqrt{m^2-1}\,\mathrm{sd}(\xi) \pm \mathrm{cd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\sqrt{2}}{\sqrt{m^2-2}}\frac{\sqrt{m^2-1}\,\mathrm{cd}(\xi)\,\mathrm{nd}(\xi) + m^2\,\mathrm{sd}(\xi)\,\mathrm{nd}(\xi) - \mathrm{sd}(\xi)\,\mathrm{nd}(\xi)}{\mathrm{sd}(\xi) + \mathrm{cd}(\xi)}\right], \tag{3.1.23}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(m^2-2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(m^2-2)(2\alpha^2-9\beta)^2}}t.$

Case 13. Choosing $P = -m^2(1-m^2), Q = 2m^2-1, R = 1$ and $G(\xi) = \mathrm{sd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\mathrm{cs}(\xi)\,\mathrm{nd}(\xi)}{\sqrt{2m^2-1}}\right], \tag{3.1.24}$$

where $\xi = \sqrt{-\frac{6\alpha^2}{(2m^2-1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2m^2-1)(2\alpha^2-9\beta)^2}}t.$

If $m \to 1$, then we have the same anti-kink soliton wave solutions (3.1.12).

Case 14. Choosing $P = \frac{m^2}{4}, Q = \frac{m^2-2}{2}, R = \frac{1}{4}$ and $G(\xi) = \text{sn}(\xi) \ / \ [1 \pm \text{dn}(\xi)]$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\sqrt{2}}{\sqrt{m^2-2}}\,\text{ns}(\xi)\,\text{cn}(\xi)\left(\text{dn}(\xi) + \frac{m^2\,\text{sn}^2(\xi)}{1+\text{dn}(\xi)}\right)\right], \tag{3.1.25}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(m^2-2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(m^2-2)(2\alpha^2-9\beta)^2}}t.$

Case 15. Choosing $P = \frac{-1}{4}, Q = \frac{m^2+1}{2}, R = \frac{(1-m^2)^2}{4}$ and $G(\xi) = m\,\text{cn}(\xi) \ \pm \text{dn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{m\sqrt{2}\,\text{sn}(\xi)}{\sqrt{m^2+1}}\right], \tag{3.1.26}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(m^2+1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(m^2+1)(2\alpha^2-9\beta)^2}}t.$

If $m \to 1$, then we have the same kink soliton wave solutions (3.1.8).

Case 16. Choosing $P = \frac{(1-m^2)^2}{4}, Q = \frac{m^2+1}{2}, R = \frac{1}{4}$ and $G(\xi) = \text{ds}(\xi) \ \pm \text{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{\sqrt{2}\,\text{ns}(\xi)}{\sqrt{m^2+1}}\right], \tag{3.1.27}$$

where $\xi = \sqrt{-\frac{12\alpha^2}{(m^2+1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(m^2+1)(2\alpha^2-9\beta)^2}}t.$

If $m \to 0$, then we have the same exact wave solutions (3.1.14).

If $m \to 1$, then we have the same anti-kink soliton wave solutions (3.1.12).

where $\xi = \sqrt{-\frac{12\alpha^2}{(m^2+1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(m^2+1)(2\alpha^2-9\beta)^2}}t.$

Case 17. Choosing $P = \frac{1}{4}, Q = \frac{m^2-2}{2}, R = \frac{m^2}{4}$ and $G(\xi) = \text{dc}(\xi) \ \pm \sqrt{1-m^2}\,\text{nc}(\xi)$, we obtain the

Jacobi elliptic function solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\sqrt{2}}{\sqrt{m^2-2}}\left(\frac{(1-m^2)\operatorname{sc}(\xi)\operatorname{nc}(\xi)+\sqrt{1-m^2}\operatorname{dc}(\xi)\operatorname{sc}(\xi)}{\operatorname{dc}(\xi)+\sqrt{1-m^2}\operatorname{nc}(\xi)}\right)\right], \tag{3.1.28}$$

where $\xi=\sqrt{-\frac{12\alpha^2}{(m^2-2)(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{108\alpha^2\beta}{(m^2-2)(2\alpha^2-9\beta)^2}}t.$

Case 18. Choosing $P=1, Q=2-4m^2, R=1$ and $G(\xi)=\operatorname{sn}(\xi)\operatorname{dn}(\xi)\ /\operatorname{cn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\operatorname{dn}(\xi)\operatorname{cs}(\xi)-m^2\operatorname{sn}(\xi)\operatorname{cs}(\xi)\operatorname{sd}(\xi)+\operatorname{sn}(\xi)\operatorname{dc}(\xi)}{\sqrt{2-4m^2}}\right], \tag{3.1.29}$$

where $\xi=\sqrt{-\frac{6\alpha^2}{(2-4m^2)(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{54\alpha^2\beta}{(2-4m^2)(2\alpha^2-9\beta)^2}}t.$

If $m\to 0$, then we have the same exact wave solutions (3.1.11).

Case 19. Choosing $P=m^4, Q=2m^2-4, R=1$ and $G(\xi)=\operatorname{sn}(\xi)\operatorname{cn}(\xi)\ /\operatorname{dn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\operatorname{cs}(\xi)\operatorname{cn}(\xi)\operatorname{dc}(\xi)-\operatorname{sn}(\xi)\operatorname{dc}(\xi)+m^2\operatorname{sd}(\xi)\operatorname{cn}(\xi)}{\sqrt{2m^2-4}}\right], \tag{3.1.30}$$

where $\xi=\sqrt{-\frac{6\alpha^2}{(2m^2-4)(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{54\alpha^2\beta}{(2m^2-4)(2\alpha^2-9\beta)^2}}t.$

Case 20. Choosing $P=1, Q=2m^2+2, R=1-2m^2+m^4$ and $G(\xi)=\operatorname{dn}(\xi)\operatorname{cn}(\xi)\ /\operatorname{sn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\operatorname{dc}(\xi)\operatorname{dn}(\xi)\operatorname{sd}(\xi)+\operatorname{cs}(\xi)\operatorname{dn}(\xi)+m^2\operatorname{sd}(\xi)\operatorname{cn}(\xi)}{\sqrt{2m^2+2}}\right], \tag{3.1.31}$$

where $\xi=\sqrt{-\frac{6\alpha^2}{(2m^2+2)(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{54\alpha^2\beta}{(2m^2+2)(2\alpha^2-9\beta)^2}}t.$

If $m\to 0$, then Eq. (1.1) has the exact wave solutions

$$V(\xi)=\frac{\alpha}{3\beta}\left[1\pm\frac{\tan(\xi)+\cot(\xi)}{\sqrt{2}}\right], \tag{3.1.32}$$

where $\xi=\sqrt{-\frac{3\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x-\sqrt{\frac{27\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t.$

If $m \to 1$, then Eq. (1.1) has the exact wave solutions

$$V(\xi) = \frac{\alpha}{3\beta}\left[1 \pm \frac{2\tanh(\xi) + \operatorname{sech}(\xi)\operatorname{csch}(\xi)}{2}\right], \tag{3.1.33}$$

where $\xi = \sqrt{-\frac{3\alpha^2}{2\ (2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{27\alpha^2\beta}{2\ (2\alpha^2-9\beta)^2}}t.$

3.2. Exact wave solutions of Eq. (1.2) using the method of Sec. 2

In this subsection, to solve Eq. (1.2) using the generalized $\left(\frac{G'}{G}\right)$ -expansion method of Sec. 2, we look for the voltage $E(x,t)$ of the pulse in the traveling form:

$$E(x,t) = E(\xi), \qquad \xi = x - vt, \tag{3.2.1}$$

where v is the propagation velocity of the pulse. Substituting (3.2.1) into Eq. (1.2), we have the following nonlinear ODE:

$$E''(\xi) + k_1 E(\xi) + k_2 E^2(\xi) = 0, \tag{3.2.2}$$

where

$$k_1 = -\frac{12(v^2 - v_0^2)}{\sigma^2 v_0^2},\ k_2 = \frac{6b_1 v^2}{\sigma^2 v_0^2}. \tag{3.2.3}$$

Balancing E'' with E^2 gives $N = 2$. Therefore, (2.4) reduces to

$$E(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{3.2.4}$$

where $a_0,\ a_1$ and a_2 are constants to be determined such that $a_2 \neq 0$.
Substituting (3.2.4) along with the Jacobi elliptic equation (2.5) into Eq. (3.2.2) and collecting all the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i$, $(i = 0, 1, ..., 4)$ and setting them to be zero, we have the following algebraic equations:

$\left(\frac{G'(\xi)}{G(\xi)}\right)^4 : a_2^2 k_2 + 6\,a_2 = 0,$

$\left(\frac{G'(\xi)}{G(\xi)}\right)^3 : a_1 a_2 k_2 + a_1 = 0,$

$\left(\frac{G'(\xi)}{G(\xi)}\right)^2 : -8\,a_2 Q + k_1 a_2 + k_2\,(2\,a_0 a_2 + a_1^2) = 0,$

$\left(\frac{G'(\xi)}{G(\xi)}\right) : 2\,a_0a_1k_2 - 2\,Qa_1 + a_1k_1 = 0,$

$$\left(\frac{G'(\xi)}{G(\xi)}\right)^0 : -8\,RPa_2+2\,a_2Q^2+k_2a_0^2+k_1a_0 = 0. \tag{3.2.5}$$

On solving the above algebraic equations (3.2.5) using the CAS, we have the following result:

$$a_0 = \frac{2\left(2Q \pm \sqrt{12PR+Q^2}\right)}{k_2}, \quad a_1 = 0, \quad a_2 = -\frac{6}{k_2}, \;\; k_1 = \mp 4\sqrt{12PR+Q^2}, \tag{3.2.6}$$

where $Q^2 + 12PR > 0$.

Substituting (3.2.6) into (3.2.4) yields

$$E(\xi) = -\frac{2}{k_2}\left[2Q \pm \sqrt{12PR+Q^2} - 3\left(\frac{G'(\xi)}{G(\xi)}\right)^2,\right], \tag{3.2.7}$$

where $\xi = x - vt$.

With reference to solving Eq. (2.5), we deduce that the Jacobi elliptic function solutions and other exact solutions of Eq. (1.2) as follows:
Case 1. Choosing $P = -m^2, Q = 2m^2 - 1, R = 1 - m^2$ and $G(\xi) = \mathrm{cn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4m^2-2\right) \pm \sqrt{16m^4 - 16m^2 + 1} - 3\,\mathrm{dc}^2(\xi)\,\mathrm{sn}^2(\xi)\right], \tag{3.2.8}$$

where $k_1 = \mp 4\sqrt{16m^4 - 16m^2 + 1}$.
If $m \to 0$, then Eq. (1.2) has the periodic wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(-2 \pm 1) - 3\tan^2(\xi)\right], \tag{3.2.9}$$

while, If $m \to 1$, then Eq. (1.2) has the kink soliton wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(2 \pm 1) - 3\tanh^2(\xi)\right], \tag{3.2.10}$$

where $k_1 = \mp 4$.
Case 2. Choosing $P = -1, Q = 2 - m^2, R = m^2 - 1$ and $G(\xi) = \mathrm{dn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4-2m^2\right) \pm \sqrt{m^4 - 16m^2 + 16} - 3m^4\,\mathrm{cd}^2(\xi)\,\mathrm{sn}^2(\xi)\right], \tag{3.2.11}$$

where $k_1 = \mp 4\sqrt{m^4 - 16m^2 + 16}$.
If $m \to 1$, then we have the same kink soliton wave solutions (3.2.10).

Case 3. Choosing $P = 1 - m^2, Q = 2 - m^2, R = 1$ and $G(\xi) = \mathrm{sc}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4 - 2m^2\right) \pm \sqrt{m^4 - 16m^2 + 16} - 3\,\mathrm{ds}^2(\xi)\,\mathrm{nc}^2(\xi)\right], \tag{3.2.12}$$

where $k_1 = \mp 4\sqrt{m^4 - 16m^2 + 16}$.
If $m \to 0$, then Eq. (1.2) has the exact wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(4 \pm 4) - 3\sec^2(\xi)\csc^2(\xi)\right], \tag{3.2.13}$$

where $k_1 = \mp 16$.
If $m \to 1$, then we have the same kink soliton wave solutions (3.2.10).

Case 4. Choosing $P = \frac{1}{4}, Q = \frac{1-2m^2}{2}, R = \frac{1}{4}$ and $G(\xi) = \mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(1 - 2m^2\right) \pm \sqrt{m^4 - m^2 + 1} - 3\,\mathrm{ds}^2(\xi)\right], \tag{3.2.14}$$

where $k_1 = \mp 4\sqrt{m^4 - m^2 + 1}$.
If $m \to 0$, then Eq. (1.2) has the exact wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(1 \pm 1) - 3\csc^2(\xi)\right], \tag{3.2.15}$$

where $k_1 = \mp 4$.
If $m \to 1$, then Eq. (1.2) has the anti-bell soliton wave solution

$$E(\xi) = \frac{2}{k_2}\left[(-1 \pm 1) - 3\,\mathrm{csch}^2(\xi)\right], \tag{3.2.16}$$

where $k_1 = \mp 4$.
Case 5. Choosing $P = -1, Q = 2 - m^2, R = m^2 - 1$ and $G(\xi) = \mathrm{nd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4 - 2m^2\right) \pm \sqrt{m^4 - 16m^2 + 16} - 3m^4\,\mathrm{cn}^2(\xi)\,\mathrm{sd}^2(\xi)\right], \tag{3.2.17}$$

where $k_1 = \mp 4\sqrt{m^4 - 16m^2 + 16}$.

If $m \to 1$, then we have the same kink soliton wave solutions (3.2.10).

Case 6. Choosing $P = 1 - m^2, Q = 2 - m^2, R = 1$ and $G(\xi) = \mathrm{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4 - 2m^2\right) \pm \sqrt{m^4 - 16m^2 + 16} - 3\,\mathrm{dc}^2(\xi)\,\mathrm{ns}^2(\xi)\right], \tag{3.2.18}$$

where $k_1 = \mp 4\sqrt{m^4 - 16m^2 + 16}$.
If $m \to 0$, then we have the same exact wave solutions (3.2.13).
If $m \to 1$, then we have the same kink soliton wave solutions (3.2.10).

Case 7. Choosing $P = m^2(m^2 - 1), Q = 2m^2 - 1, R = 1$ and $G(\xi) = \mathrm{ds}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4m^2 - 2\right) \pm \sqrt{16m^4 - 16m^2 + 1} - 3\,\mathrm{cd}^2(\xi)\,\mathrm{ns}^2(\xi)\right], \tag{3.2.19}$$

where $k_1 = \mp 4\sqrt{16m^4 - 16m^2 + 1}$.
If $m \to 0$, then Eq. (1.2) has the periodic wave solutions

$$E(\xi) = \frac{-2}{k_2}\left[(2 \mp 1) + 3\cot^2(\xi)\right], \tag{3.2.20}$$

where $k_1 = \mp 4$.
If $m \to 1$, then Eq. (1.2) has the anti-kink soliton wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(2 \pm 1) - 3\coth^2(\xi)\right], \tag{3.2.21}$$

where $k_1 = \mp 4$.
Case 8. Choosing $P = \frac{m^2-1}{4}, Q = \frac{m^2+1}{2}, R = \frac{m^2-1}{4}$ and $G(\xi) = m\,\mathrm{sd}(\xi) \pm \mathrm{nd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(m^2 + 1\right) \pm \sqrt{m^4 - m^2 + 1} - 3m^2\,\mathrm{cd}^2(\xi)\right], \tag{3.2.22}$$

where $k_1 = \mp 4\sqrt{m^4 - m^2 + 1}$.

Case 9. Choosing $P = \frac{1-m^2}{4}, Q = \frac{m^2+1}{2}, R = \frac{1-m^2}{4}$ and $G(\xi) = \mathrm{nc}(\xi) \pm \mathrm{sc}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(1 + m^2\right) \pm \sqrt{m^4 - m^2 + 1} - 3\,\mathrm{dc}^2(\xi)\right], \tag{3.2.23}$$

where $k_1 = \mp 4\sqrt{m^4 - m^2 + 1}$.
If $m \to 0$, then we have the same exact wave solutions (3.2.15).
If $m \to 1$, then we have the same kink soliton wave solutions (3.2.10).

Case 10. Choosing $P = \frac{m^2}{4}, Q = \frac{m^2-2}{2}, R = \frac{m^2}{4}$ and $G(\xi) = \sqrt{m^2-1}\,\mathrm{sd}(\xi) \pm \mathrm{cd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(m^2-2\right) \pm \sqrt{m^4 - m^2 + 1} - 3\frac{(m^2\,\mathrm{sn}(\xi) + \sqrt{m^2-1}\,\mathrm{cn}(\xi) - \mathrm{sn}(\xi))^2}{\mathrm{dn}^2(\xi)(\sqrt{m^2-1}\,\mathrm{sn}(\xi) + \mathrm{cn}(\xi))^2}\right], \tag{3.2.24}$$

where $k_1 = \mp 4\sqrt{m^4 - m^2 + 1}$.
Case 11. Choosing $P = -m^2(1-m^2), Q = 2m^2-1, R = 1$ and $G(\xi) = \mathrm{sd}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4m^2-2\right) \pm \sqrt{16m^4 - 16m^2 + 1} - 3\,\mathrm{cs}^2(\xi)\,\mathrm{nd}^2(\xi)\right], \tag{3.2.25}$$

where $k_1 = \mp 4\sqrt{16m^4 - 16m^2 + 1}$.
If $m \to 0$, then we have the same periodic wave solutions (3.2.20).
If $m \to 1$, then we have the same anti-kink soliton wave solutions (3.2.21).

Case 12. Choosing $P = \frac{(1-m^2)^2}{4}, Q = \frac{m^2+1}{2}, R = \frac{1}{4}$ and $G(\xi) = \mathrm{ds}(\xi) \pm \mathrm{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(m^2+1\right) \pm \sqrt{m^4 - m^2 + 1} - 3\,\mathrm{ns}^2(\xi)\right], \tag{3.2.26}$$

where $k_1 = \mp 4\sqrt{m^4 - m^2 + 1}$.
If $m \to 0$, then we have the same exact wave solutions (3.2.15).

If $m \to 1$, then we have the same anti-kink soliton wave solutions (3.2.21).

Case 13. Choosing $P = 1, Q = 2 - 4m^2, R = 1$ and $G(\xi) = \mathrm{sn}(\xi)\,\mathrm{dn}(\xi)\ /\,\mathrm{cn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4-8m^2\right) \pm 4\sqrt{m^4 - m^2 + 1} - 3\frac{(-2m^2\,\mathrm{sn}^2(\xi) + m^2\,\mathrm{sn}^4(\xi) + 1)^2}{\mathrm{cn}^2(\xi)\,\mathrm{sn}^2(\xi)\,\mathrm{dn}^2(\xi)}\right], \tag{3.2.27}$$

where $k_1 = \mp 16\sqrt{m^4 - m^2 + 1}$.
If $m \to 0$, then Eq. (1.2) has the exact wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(4 \pm 4) - 3(\tan(\xi) + \cot(\xi))^2\right], \tag{3.2.28}$$

where $k_1 = \mp 16$.
If $m \to 1$, then Eq. (1.2) has the exact wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(-4 \pm 4) - 3\,\mathrm{sech}^2(\xi)\,\mathrm{csch}^2(\xi)\right], \tag{3.2.29}$$

where $k_1 = \mp 16$.
Case 14. Choosing $P = m^4, Q = 2m^2 - 4, R = 1$ and $G(\xi) = \mathrm{sn}(\xi)\,\mathrm{cn}(\xi)\ /\,\mathrm{dn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4m^2 - 8\right) \pm 4\sqrt{m^4 - m^2 + 1} - 3\frac{(m^2\,\mathrm{sn}^4(\xi) - 2\,\mathrm{sn}^2(\xi) + 1)^2}{\mathrm{dn}^2(\xi)\,\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)}\right], \tag{3.2.30}$$

where $k_1 = \mp 16\sqrt{m^4 - m^2 + 1}$.
If $m \to 0$, then Eq. (1.2) has the exact wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(-8 \pm 4) - 3(\cot(\xi) - \tan(\xi))^2\right], \tag{3.2.31}$$

where $k_1 = \mp 16$.
If $m \to 1$, then we have the exact wave solutions (3.2.29).

Case 15. Choosing $P = 1, Q = 2m^2 + 2, R = 1 - 2m^2 + m^4$ and $G(\xi) = \mathrm{dn}(\xi)\,\mathrm{cn}(\xi)\ /\,\mathrm{sn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4m^2 + 4\right) \pm 4\sqrt{m^4 - m^2 + 1} - 3\frac{(m^2\,\mathrm{sn}^4(\xi) - 1)^2}{\mathrm{sn}^2(\xi)\,\mathrm{cn}^2(\xi)\,\mathrm{dn}^2(\xi)}\right], \tag{3.2.32}$$

where $k_1 = \mp 16\sqrt{m^4 - m^2 + 1}$.
If $m \to 0$, then we have the same exact wave solutions (3.2.28).
If $m \to 1$, then Eq. (1.2) has the exact wave solutions

$$E(\xi) = \frac{2}{k_2}\left[(8 \pm 4) - 3(\tanh(\xi) + \coth(\xi))^2\right], \tag{3.2.33}$$

where $k_1 = \mp 16$.
Case 16. Choosing $P = m^2, Q = -(1 + m^2), R = 1$ and $G(\xi) = \mathrm{sn}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(-2 - 2m^2\right) \pm \sqrt{m^4 + 14m^2 + 1} - 3\,\mathrm{cs}^2(\xi)\,\mathrm{dn}^2(\xi)\right], \tag{3.2.34}$$

where $k_1 = \mp 4\sqrt{m^4 + 14m^2 + 1}$.
If $m \to 0$, then we have the same periodic wave solutions (3.2.20).

If $m \to 1$, then we have the same exact wave solutions (3.2.29).

Case 17. Choosing $P = 1, Q = -(1+m^2), R = m^2$ and $G(\xi) = \mathrm{ns}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(-2-2m^2\right) \pm \sqrt{m^4+14m^2+1} - 3\,\mathrm{cn}^2(\xi)\,\mathrm{ds}^2(\xi)\right], \tag{3.2.35}$$

where $k_1 = \mp 4\sqrt{m^4+14m^2+1}$.
If $m \to 0$, then we have the same periodic wave solutions (3.2.20).
If $m \to 1$, then we have the same exact wave solutions (3.2.29).

Case 18. Choosing $P = 1-m^2, Q = 2m^2-1, R = -m^2$ and $G(\xi) = \mathrm{nc}(\xi)$, we obtain the Jacobi elliptic function solutions

$$E(\xi) = \frac{2}{k_2}\left[\left(4m^2-2\right) \pm \sqrt{16m^4-16m^2+1} - 3\,\mathrm{dn}^2(\xi)\,\mathrm{sc}^2(\xi)\right], \tag{3.2.36}$$

where $k_1 = \mp 4\sqrt{16m^4-16m^2+1}$.
If $m \to 0$, then we have the same periodic wave solutions (3.2.9).
If $m \to 1$, then we have the same kink soliton wave solutions (3.2.10).

1 Physical explanations of the results

We have shown in section 3, that the exact solutions of the higher order nonlinear PDEs (1.1) and (1.2) are written in terms of the Jacobi elliptic functions. Also, we have found spetial solutions from the Jacobi elliptic function solutions when the modulus $m = 1$ and $m = 0$. These solutions are kink, anti-kink shaped soliton solutions, bell-shaped soliton solutions, anti bell-shaped soliton solutions, hyperbolic solutions and periodic solutions. In this section, we will present some graphs of these solutions by choosing suitable values of the parameters to visualize the mechanism of the original nonlinear PDEs. Using mathematical software CAS, we organize these graphs as follows: In Fig. 1, the plots of the solutions (3.1.7) are drawn by choosing $\alpha = \beta = \delta = 1$, the modulus $m = 0.99$ and $m = 1$ which is the kink soliton wave solutions (3.1.7). In Fig. 2 , the plots of the solutions (3.1.20) are drawn by choosing $\alpha = 1,\ \beta = \delta = 2$, the modulus $m = 0.1$ and $m = 0$ which is the exact wave solutions (3.1.20). In Fig. 3, the plot of the solutions (3.2.27) is drawn by choosing $k_2 = -1,\ v = 1$, the modulus $m = 0.5$, $m = 1$ which is the hyperbolic wave solutions (3.2.29) and $m = 0$ which is the periodic wave solutions (3.2.28). All these figures are new and not found elsewhere, which include the graphs of the Jacobi elliptic functions.

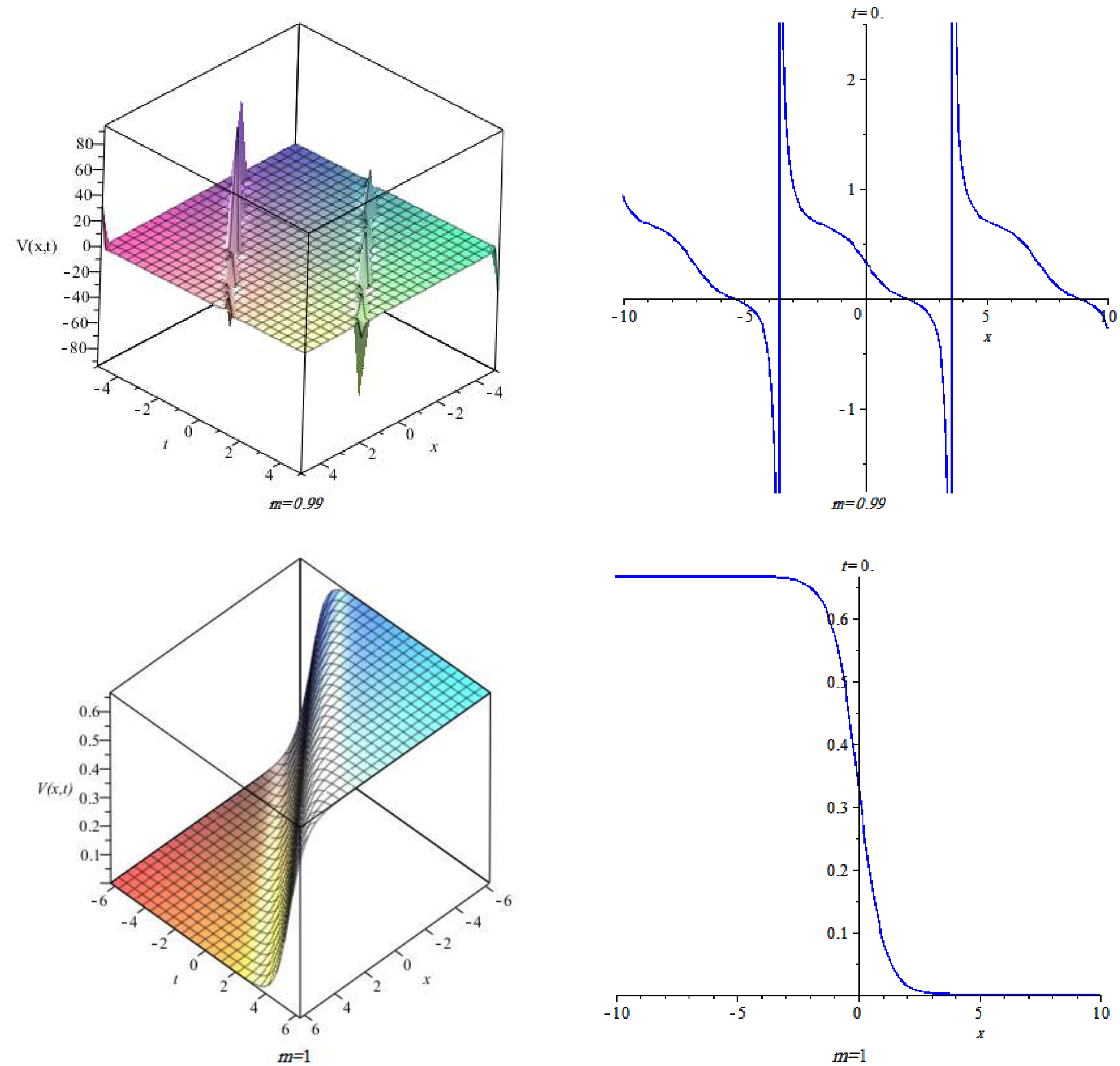


Fig. 1. Plots of the solutions (3.1.7) with $\alpha = \beta = \delta = 1,\ m = 0.99,\ 1.$

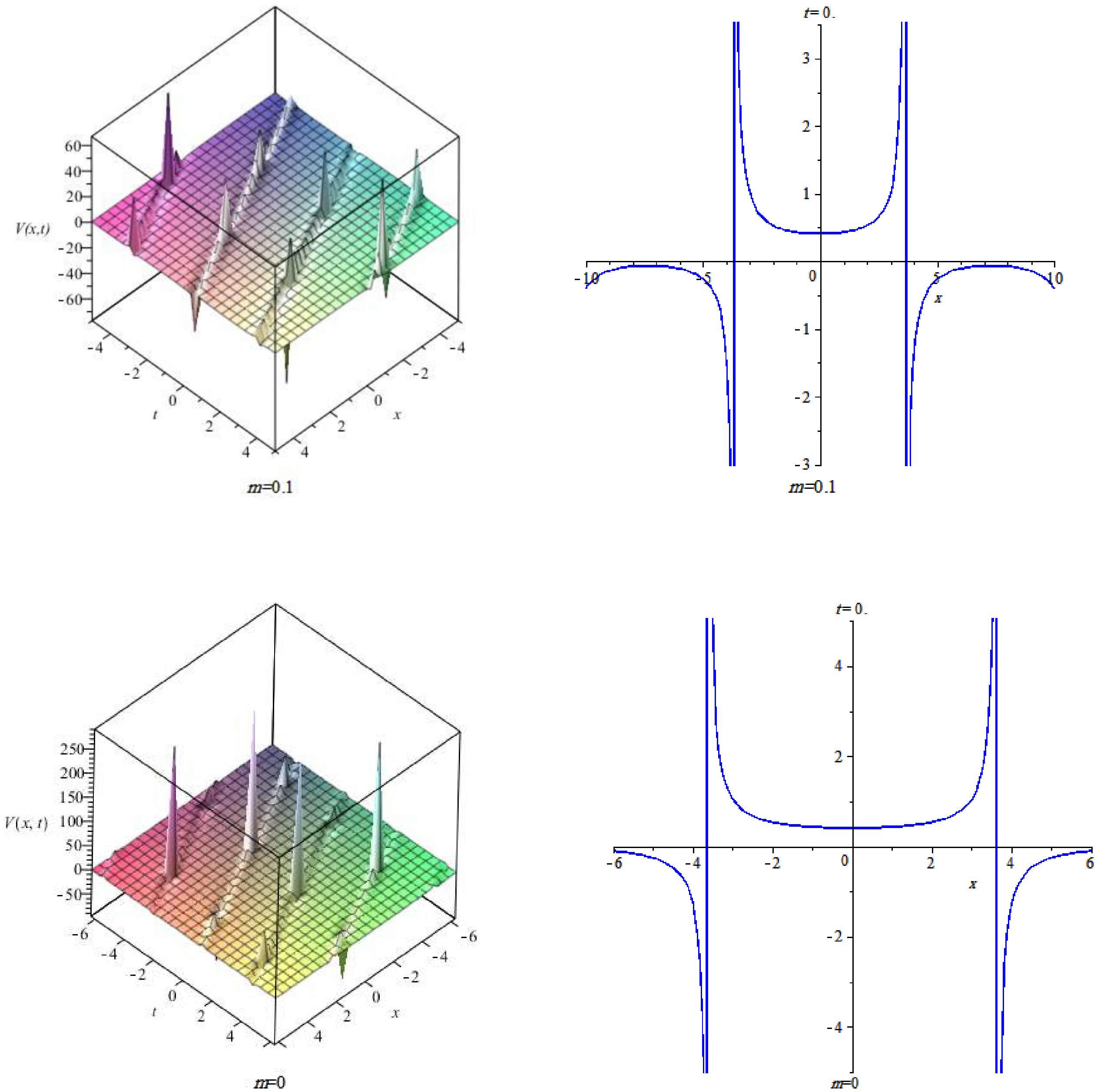
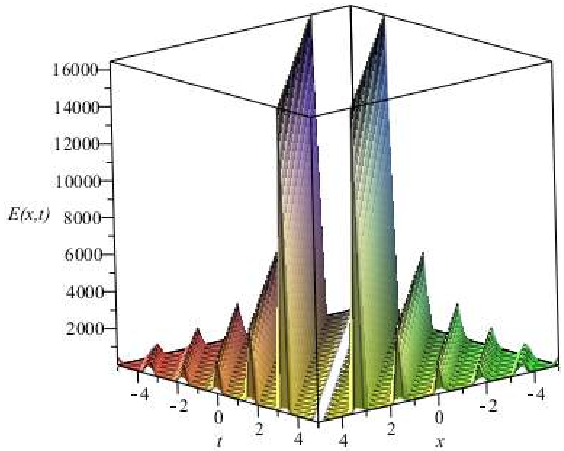
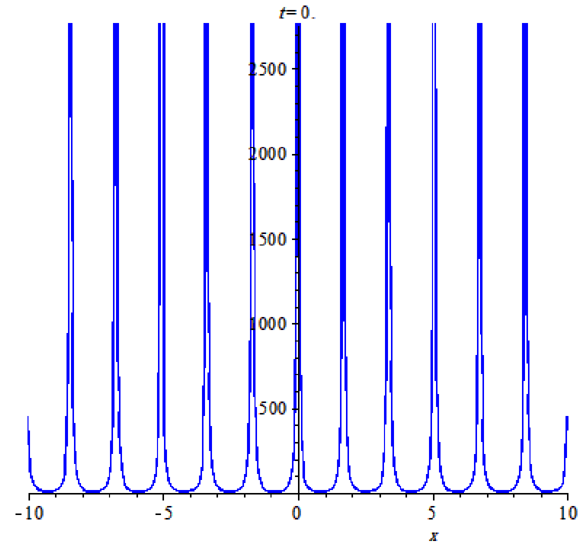


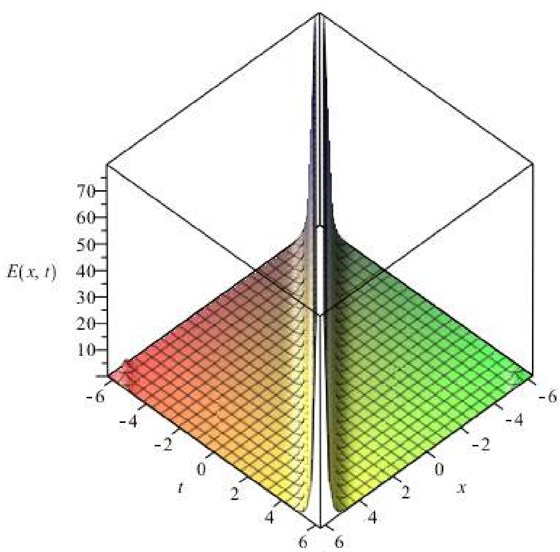
Fig. 2. Plots of the solutions (3.1.20) with $\alpha = 1,\ \beta = \delta = 2,\ m = 0.1,\ 0.$


16000
14000
12000
10000
E(x,t) 8000
6000
4000
2000
-4 -2 0 2 4
t
4 2 0 -2 -4
x

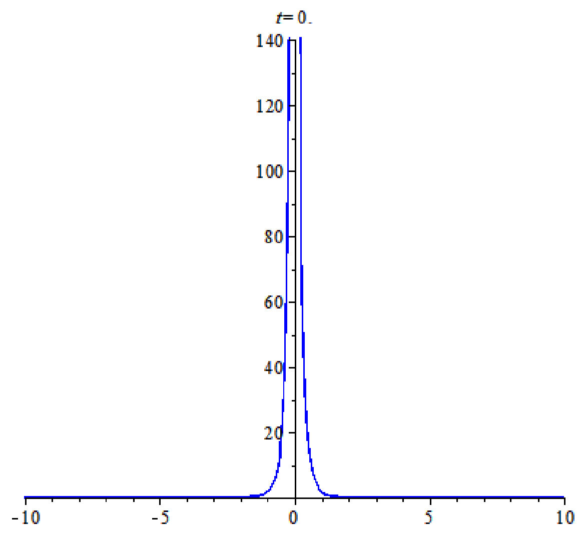
m=0.5


t= 0.
2500
2000
1500
1000
500
-10 -5 0 5 10
x

m=0.5


70
60
50
E(x, t) 40
30
20
10
-6 -4 -2 0 2 4 6
t
-6 -4 -2 0 2 4 6
x

m=1


t= 0.
140
120
100
80
60
40
20
-10 -5 0 5 10
x

m=1

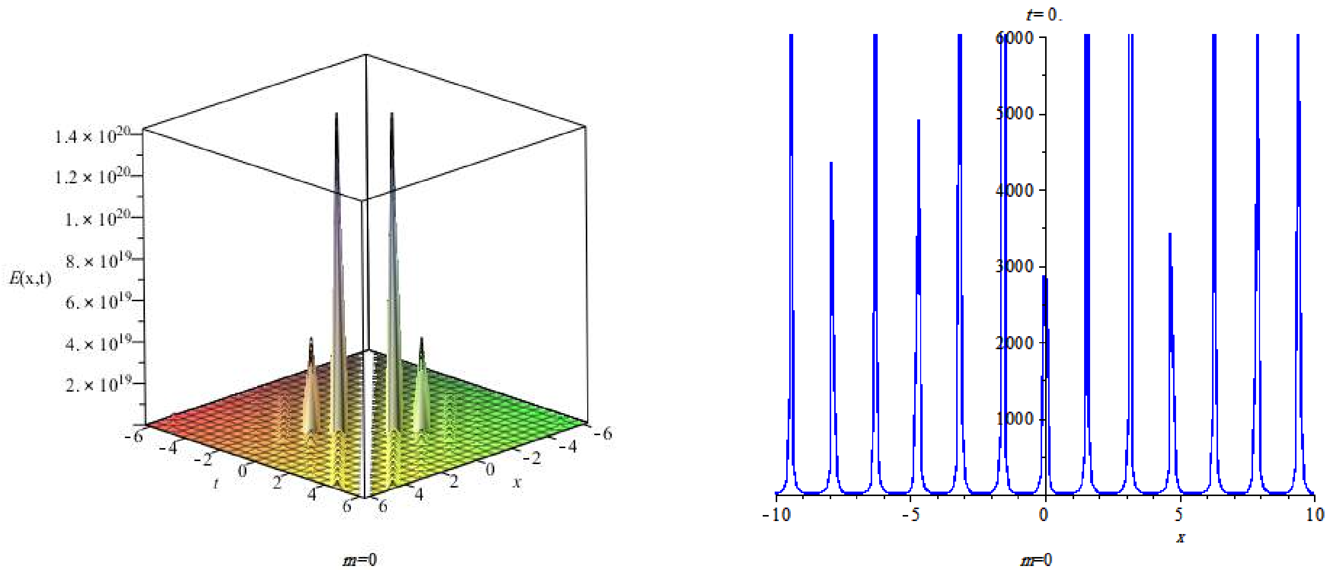


Fig. 3. Plots of the solutions (3.2.27) with $k_2 = -1,\ v = 1,\ m = 0.5,\ 1,\ 0.$

5. Conclusions.

This article, based on the generalized $(G\prime/G)$-expansion method described in Section 2 with the aid of symbolic computation CAS, we have obtained many new types of Jacobi elliptic function solutions for the two higher-order nonlinear PDEs (1.1) and (1.2) describing the nonlinear low-pass electrical lines and pulse narrowing nonlinear transmission lines respectively. Further, we have found other exact solutions of the nonlinear PDEs (1.1) and (1.2) when the modulus of the Jacobi elliptic functions m takes $1,\ 0$. these solutions include, kink and anti-kink soliton wave solutions, bell (bright) and anti-bell (dark) soliton wave solutions and periodic solutions. Comparing our results in this article with the well-known results of [23, 24, 32, 34], we conclude that our results are different and new, which are not found elsewhere. We notice that solutions obtained though the generalized $(G\prime/G)$-expansion method here are more general. The proposed method of this article is effective and can be applied to many other nonlinear PDEs. Finally, all solutions obtained in this article have been checked with the CAS by putting them back into the original equations (1.1) and (1.2).

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